On the treatment of the geodetic boundary value problem by means of regular gravity space formulations

Proof of concept based on numerical closed-loop simulations

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by

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Contents

List of important symbols, constants and abbrevations

Δ	... Laplace operator
∇	... gradient operator
Ω_x^+	... exterior domain in ordinary space
Ω_x^-	... interior domain in ordinary space
Ω_g^+	... exterior domain in gravity space
Ω_g^-	... interior domain in gravity space
σ	... physical Earth surface
Σ, Σ_g	... boundary surface in gravity space, (gravimetric) telluroid surface
S	... mean Earth sphere
W	... Earth's gravity potential
$w = W\big\vert_\sigma$	... restriction of the Earth's gravity potential onto the surface of the Earth
W_0	... normal gravity potential
$\delta W = W - W_0$	... disturbing gravity potential
$\delta w = \delta W\big\vert_\sigma$	... gravity potential disturbance; restriction of the disturbing gravity potential onto the surface of the Earth
Δw	... gravity potential anomaly
$\mathbf{\Gamma}$	... gravity vector
Γ	... modulus of the gravity vector or simply *gravity*
$\tilde{\mathbf{\Gamma}} = \mathbf{\Gamma}\big\vert_\sigma$	... restriction of the gravity vector onto the surface of the Earth
$\mathbf{\Gamma}_0$	... normal gravity vector
Γ_0	... modulus of the normal gravity vector or simply *normal gravity*
$\tilde{\mathbf{\Gamma}}_0 = \mathbf{\Gamma}_0\big\vert_\sigma$	... restriction of the normal gravity vector onto the surface of the Earth
$\delta\mathbf{\Gamma}$	... gravity disturbance vector
$\delta\Gamma$	... (scalar) gravity disturbance
$\delta\tilde{\mathbf{\Gamma}} = \delta\mathbf{\Gamma}\big\vert_\sigma$	... restriction of the gravity disturbance vector onto the surface of the Earth
$\Delta\mathbf{\Gamma}$	... gravity anomaly vector
$\Delta\Gamma$	... (scalar) gravity anomaly
$\Delta\Gamma'$	... component of the gravity anomaly vector in the direction of the isozenithal
Z	... Earth's centrifugal potential
$\mathbf{a}_z$	... centrifugal acceleration vector
ω	... angular velocity of the Earth rotation
V	... Earth's gravitational potential
$v = V\big\vert_\sigma$	... restriction of the Earth's gravitational potential onto the surface of the Earth
V_0	... normal gravitational potential
$\delta V = V - V_0$	... disturbing gravitational potential; ($\delta V = \delta W$, if $W_0 = V_0 + Z$ holds)
$\delta v = \delta V\big\vert_\sigma$	... gravitational potential disturbance; restriction of the disturbing gravitational potential onto the surface of the Earth
Δv	... gravitational potential anomaly
$\mathbf{g}$	... gravitational acceleration vector
g	... modulus of the gravitational acceleration vector
$\tilde{\mathbf{g}} = \mathbf{g}\big\vert_\sigma$	... restriction of the gravitational acceleration vector onto the surface of the Earth
$\mathbf{g}_0$	... normal gravitational acceleration vector
g_0	... modulus of the normal gravitational acceleration vector
$\tilde{\mathbf{g}}_0 = \mathbf{g}_0\big\vert_\sigma$	... restriction of the normal gravitational acceleration vector onto the surface of the Earth
$\delta\mathbf{g}$	... gravitational disturbance vector
$\delta\tilde{\mathbf{g}} = \delta\mathbf{g}\big\vert_\sigma$	... restriction of the gravitational disturbance vector onto the surface of the Earth
$\Delta\mathbf{g}$	... gravitational anomaly vector

ψ	... adjoint potential
ψ_0	... adjoint normal potential
$\delta\psi = \psi - \psi_0$	... adjoint disturbing potential
$\delta\tilde{\psi} = \delta\psi\big\|_\Sigma$	... adjoint potential disturbance; restriction of the adjoint disturbing potential onto the gravimetric telluroid
$\boldsymbol{\xi}$	... gravity space coordinate vector
ξ	... modulus of the gravity space coordinate vector
$\boldsymbol{\zeta}$	... vectorial height anomaly, i.e. separation topography-telluroid
ζ	... (scalar) height anomaly
τ	... arc length of an isozenithal line
ϕ, λ	... spherical latitude and longitude
h	... height above the sphere
$\bar{\phi}, \bar{\lambda}$	... geodetic latitude and longitude
$\bar{h}$	... height above the ellipsoid
Φ, Λ	... astronomical latitude and longitude
$\hat{\phi}, \hat{\lambda}$	... normal latitude and longitude
$\mathbf{J}$	... Jacobian matrix
$\mathbf{M}$	... Hessian matrix of the gravity potential W
$\mathbf{M}_0$	... Hessian matrix of the normal gravity potential W_0
$\mathbf{V}$	... Hessian matrix of the gravitational potential V
$\mathbf{V}_0$	... Hessian matrix of the normal gravitational potential V_0
$\boldsymbol{\Phi}$	... functional matrix, which corresponds to $\mathbf{V}$ in regular gravity space
$\alpha_{ij}, \beta_{ij}, \gamma_{ijk}$	... coefficients associated with $\boldsymbol{\Phi}$
φ_{ij}	... adjoint matrix elements
$\bar{\boldsymbol{\Phi}}$	... functional matrix, which corresponds to $\mathbf{V}$ in ellipsoidal regular gravity space
$\bar{\alpha}_{ij}, \bar{\beta}_{ij}, \bar{\gamma}_{ijk}$	... coefficients associated with $\bar{\boldsymbol{\Phi}}$
$\bar{\varphi}_{ij}$	... adjoint matrix elements
N	... geoidal height or geoidal undulation
GM	... product of the gravitational constant G and the Earth's mass M
R	... mean radius of the Earth's sphere
a, b	... semi-major and semi-minor axis of the Earth's ellipsoid
e	... first numerical eccentricity of the Earth's ellipsoid
J_2	... flattening of the Earth's ellipsoid at the poles
ω	... mean angular velocity of the Earth
ADC	... Analytical Data Continuation
BVP	... Boundary Value Problem
CAS	... Computer Algebra System
CoM	... Center of Mass
FFT	... Fast Fourier Transform
GBVP	... Geodetic Boundary Value Problem
GPM	... Global Potential Model
GPS	... Global Positioning System
GSA	... Gravity Space Approach
GSHA	... Global Spherical Harmonic Analysis
GSHC	... Global Spherical Harmonic Computation
GSHS	... Global Spherical Harmonic Synthesis
GRM	... Global Reference Model
GTM	... Global Topography Model

Abstract

The aim of this thesis is to present an alternative for the solution of a fundamental problem of geodesy. This problem, the so-called classical geodetic boundary value problem, comprises the determination of the figure of the Earth as well as the recovery of the Earth's gravity field in the exterior of the terrestrial masses. Already in 1849, G.G. Stokes addressed the problem of finding the Earth's gravity potential together with the physical shape of the Earth, i.e. the geoid. Later on in 1962, M.S. Molodensky proposed his famous theory for the direct gravimetric determination of the Earth's topographical surface along with the external gravity potential. Both approaches solve the initially nonlinear free boundary value problem, which implies considerable mathematical difficulties in the investigation of its existence and uniqueness properties, by means of a twofold linearization strategy. For this purpose, adequate approximations for the solution of the physical problem component, i.e. the determination of the gravity field, and for the geometrical part, i.e. the determination of the shape of the Earth's body, must be assumed. In detail, a normal potential to approximate the true potential as well as a reference surface for the geoid or the topography is required.
In 1977, F. Sansò found an elegant approach to solve the geodetic boundary value problem by transforming it from the ordinary or geometry space into a dual space. This auxiliary space is generally referred to as gravity space. F. Sansò's break-through idea is based on the application of Legendre's transformation, a member of the family of contact transformations, to obtain the corresponding boundary value problem in the newly introduced gravity space. In contrast to the conventionally treated problem, the boundary value problem in gravity space relies on a fixed boundary. Naturally, such a situation is preferable from the mathematical point of view. Remaining only is the necessity to find a suitable linearization procedure for the gravity potential determination.
Nevertheless, F. Sansò's transformed problem still suffered from a distinct singularity at the origin. Due to this reason, W. Keller encouraged the use of a modified contact transformation in 1987, which provided a fixed boundary value problem in gravity space free of any singularities. Moreover, W. Keller's revised theory succeeded to also overcome several other shortcomings of F. Sansò's gravity space transformation. Thus, in the framework of this work the terminology *regular gravity space formulations* is applied for the newly elaborated class of gravity space approaches related to the methodology pioneered by W. Keller. Indeed, W. Keller's concept additionally benefits from the fact that in its linearized version the resulting boundary value problem in dual space is analogous to the one of the simple Molodensky problem, which results within the scope of the classical solution procedure. This agreement clearly allows for making further use of all computing tools currently available for the well-established procedure of solving the classical Molodensky problem. It remains an open question why the regular gravity space concept has not yet been implemented numerically despite its obvious conceptual advantages.

Hence, after setting the classical theory, F. Sanò's approach and the two new regular formulations in contrast with each other, thereby introducing the basic theoretical principles and discussing the benefits and drawbacks of the particular methods, this work aims for the first time at the systematic numerical implementation still outstanding for the latest two regular approaches. Leadoff numerical experiments within a specially drafted closed-loop study based on a global simulation scenario indicate that the determination of the Earth's surface and of the terrestrial gravity field can also successfully be carried out in regular gravity space. However, the respective initial results suggest that the linearization of the geodetic boundary value problem in regular gravity space with respect to an isotropic linearization point, as is commonly assumed in the contributions of F. Sansò and W. Keller, is insufficient. In this context, typical closed-loop errors for the geometrical problem part, i.e. the determination of the Earth's surface, in the range of several meters apply. Theoretical as well as computational deficiencies must be made responsible for this inaccuracy as will be outlined in detail.

Consequently, this dissertation further aims to examine the suitability of using a more sophisticated linearization process. Following the familiar example of the Somigliana-Pizzetti type of normal potential, the application of an ellipsoidal normal potential or, more precisely, spheroidal normal potential is intended for the linearization procedure. The key questions to be answered are whether the overall formulae work and, in particular, the mathematical structure of a simple Molodensky problem can be preserved and whether introducing a spheroidal linearization point is also numerically advantageous.

As a matter of fact, giving up the demand to represent the transformation formulae between ordinary and auxiliary space in a closed form, which is indeed possible for the approaches according to F. Sansò and W. Keller, and adopting a series expansion representation for the underlying transformation relations instead, allows for maintaining the general formalism. That is, the designated conformity with the simple Molodensky problem continues to exist. In addition, the numerical outcome improves as expected. In short, the error level scales from several meters down to the decimeter level in case of employing the revised linearization concept. Actually, the remaining error of less than half a meter still partly results as a consequence of the necessity to truncate the series representation adopted for the transformation relations. On this account, at least theoretically, the associated error could be minimized by further expanding the corresponding series.

In summary it can be stated that the present work not only attends at length to the central issues of solving the well-known geodetic boundary value problem analytically but also provides a set of applicable numerical methods. The conducted numerical experiments document the successful accomplishment of the intended proof of concept for the approaches devised in regular gravity space. Consequently, the general applicability of the regular gravity space formulations proposed here is verified. All the same, it should be pointed out that the focus of the thesis has been rather on the expansion of the theory of the geodetic boundary value problem together with the elimination of various theoretical weak points than on the refinement of the already existing computational tools.

Zusammenfassung

Ziel dieser Dissertation ist es einen alternativen Lösungsweg für eines der grundlegendsten Probleme der Geodäsie zu erarbeiten. Dieses Problem, auch bezeichnet als klassische geodätische Randwertaufgabe, besteht darin, die Gestalt der Erde sowie das Erdschwerefeld im massefreien Außenraum der Erde zu bestimmen. Bereits um 1849 widmete sich G.G. Stokes der Bestimmung des Schwerepotentials im Erdaußenraum sowie der Festlegung einer physikalisch bedeutsamen Approximation der Erdfigur, dem so genannten Geoid. Im Jahre 1962, und somit einige Zeit später, stellte M.S. Molodensky seine mittlerweile berühmt gewordene Theorie zur direkten gravimetrischen Bestimmung der Erdoberfläche und des äußeren Schwerepotentials vor. Beide Ansätze lösen das ursprüngliche nichtlineare freie Randwertproblem, welches hinsichtlich der Untersuchung von Existenz- und Eindeutigkeitseigenschaften eine besonders hohe mathematische Komplexität aufweist, mittels einer zweifachen Linearisierungsstrategie. Zu diesem Zweck müssen geeignete Näherungslösungen für die physikalische Problemkomponente, d.h. für die Bestimmung des Schwerefeldes, wie auch für den geometrischen Problemteil, also für die Bestimmung der Gestalt des Erdkörpers, angenommen werden. Im Einzelnen bedeutet dies, dass sowohl ein Normalpotential zur Approximation des tatsächlichen Potentials als auch eine Referenzfläche für das Geoid beziehungsweise für die Topographie benötigt wird.
F. Sansò fand 1977 einen eleganten Ansatz, die geodätische Randwertaufgabe durch Transformation aus dem gewöhnlichen Anschauungs- oder Geometrieraum in einen Dualraum zu lösen. Üblicherweise wird dieser Raum als Schwereraum bezeichnet. Die bahnbrechende Idee F. Sansò's basiert auf der Verwendung der Legendre-Transformation, einer Transformation aus der Klasse der Kontakttransformationen, um das zugehörige Randwertproblem bezüglich des neu eingeführten Schwereraum zu erhalten. Im Gegensatz zum herkömmlich betrachteten Problem beruht das Randwertproblem im Schwereraum allerdings auf einer bekannten Randfläche. Selbstverständlich ist ein solcher Sachverhalt aus mathematischer Sicht der Dinge wünschenswert. Somit verbleibt ausschließlich die Notwendigkeit, eine passende Methode zur Linearisierung des Problems der Bestimmung des Schwerepotentials zu finden.
Dessen ungeachtet leidet das nach F. Sansò transformierte Problem aber nach wie vor unter einer ausgeprägten Singularität im Ursprung. Aus diesem Grund regte W. Keller im Jahre 1987 den Einsatz einer modifizierten Kontakttransformation an, die zu einem Randwertproblem des Schwereraums ohne jegliche Singularitäten führt. Darüber hinaus gelang es mit der überarbeiteten Theorie nach W. Keller auch einige andere Mängel der F. Sansò'schen Schwereraumtransformation zu überwinden. Folglich wird im Rahmen dieser Arbeit die Terminologie *reguläre Schwereraumformulierungen* für die erstmals ausführlich ausgearbeiteten Schwereraumansätze, die auf der von W. Keller bereiteten Methodik begründet sind, verwendet. Tatsächlich profitiert W. Keller's Konzept zusätzlich von dem Umstand, dass das in linearisierter Form resultierende Randwertproblem des Dualraumes mit dem Randwertproblem übereinstimmt, das dem einfachen Molodensky-Problem im Rahmen der klassischen Lösung zu Grunde liegt. Dieser Übereinstimmung ist es zu verdanken, dass alle für die gängige Vorgehensweise zur Lösung des klassischen Molodensky'schen Problems zur Verfügung stehenden rechentechnischen Werkzeuge weiter verwendet werden können. Die Frage, warum das reguläre Schwereraumkonzept trotz dessen offensichtlicher Vorteile zuvor noch nicht numerisch implementiert wurde, bleibt unbeantwortet.

Nach einer vergleichenden Gegenüberstellung der klassischen Theorie, des F. Sansò'schen Ansatzes und der beiden neuen regulären Formulierungen, wobei ferner die grundlegenden theoretischen Zusammenhänge vorgestellt und die Vor- und Nachteile der jeweiligen Ansätze diskutiert werden, ist es folglich das Ziel dieser Arbeit, erstmals eine systematische numerische Umsetzung, die für die neu hinzugekommenen regulären Ansätze noch aussteht, vorzunehmen. Erste numerische Experimente im Rahmen einer eigens dafür entworfenen globalen Studie mit geschlossenem Simulationskreislauf lassen vermuten, dass die Bestimmung der Erdoberfläche und des Erdschwerefeldes auch im regulären Schwereraum erfolgreich durchgeführt werden kann. Allerdings deuten diese einführenden Ergebnisse auch darauf hin, dass für die Linearisierung des geodätischen Randwertproblems in der regulären Schwereraumformulierung ein isotroper Linearisierungspunkt, wie er in den Arbeiten von F. Sansò und W. Keller in der Regel angenommen wird, nicht ausreichend ist. In diesem Zusammenhang sind typische Schleifenschlussfehler für den geometrischen Problemteil, d.h. für die Bestimmung der Erdoberfläche, in der Größenordnung von einigen Metern zu nennen. Unzulänglichkeiten theoretischer wie auch rechentechnischer Natur sind für diese Ungenauigkeit verantwortlich, worauf ausführlich eingegangen wird.

Dementsprechend ist es ein weiteres Anliegen dieser Dissertation die Verwendung eines verfeinerten Linearisierungsansatzes auf seine Eignung hin zu untersuchen. Dem Beispiel des bekannten Normalpotentials nach Somigliana-Pizzetti folgend ist der Einsatz eines ellipsoidischen, genauer gesagt eines sphäroidischen Normalpotentials, für den Linearisierungsschritt vorgesehen. Die zu klärende Schlüsselfrage wird sein, ob in diesem Fall der Formelapparat in seiner Gesamtheit und insbesondere die Struktur des einfachen Molodenky'schen Problems erhalten werden kann, und ob sich die Einführung eines sphäroidischen Linearisierungspunktes überhaupt numerisch vorteilhaft auswirkt.

Wird die Forderung aufgegeben, die Transformationsbeziehungen zwischen Geometrie- und Dualraum in geschlossener Form darzustellen, was für die Ansätze von F. Sansò und W. Keller gelingt, ist es in der Tat möglich durch den Übergang zu einer Reihendarstellung den generellen Formalismus beizubehalten. Das heißt, die gewünschte Übereinstimmung mit dem einfachen Problem nach Molodensky existiert weiter. Hinzu kommt, dass sich auch die numerischen Ergebnisse verbessern. Kurzum, mit der Nutzung der überarbeiteten Linearisierungsmethode sinkt das Fehlerniveau von einigen Metern auf Größenordnungen im Dezimeterbereich ab. Tatsächlich resultiert der unter einem halben Meter liegende Restfehler auch teilweise aus dem Umstand, dass die für die Transformationsbeziehung gewählte Reihendarstellung abgebrochen wird. Aus diesem Grund kann der zugehörige Fehler, zumindest rein theoretisch, durch eine Ausweitung der Reihenentwicklung noch verringert werden.

Zusammenfassend lässt sich sagen, dass die vorliegende Arbeit nicht nur umfassend die zentralen Gesichtspunkte zur analytischen Lösung der wohlbekannten geodätischen Randwertaufgabe behandelt, sondern auch eine Zusammenstellung geeigneter numerischer Methoden mit an die Hand gibt. Die durchgeführten numerischen Versuche belegen die erfolgreiche Umsetzung der angestrebten Machbarkeitsstudie für die im regulären Schwereraum geltenden Ansätze. Somit ist die generelle Anwendbarkeit der hier angeregten regulären Schwereraumformulierungen nachgewiesen. Gleichwohl sollte darauf hingewiesen werden, dass der Schwerpunkt dieser Arbeit eher auf der Erweiterung und dem Schließen einer Lücke im theoretischen Fundament des geodätischen Randwertproblems liegt als auf einer Weiterentwicklung bereits existierender rechentechnischer Werkzeuge.

Chapter 1

Introduction

1-1 Geodetic research in historical perspective

The controversy about the figure of the Earth in the time from about 1660 until 1750 may be considered as the foundation of modern geodesy, see e.g. [3],[4] Bialas 1972,1982. Prior to this, the accomplishment of ancient and medieval geodesy can be briefly summarized as follows. The knowledge about the spherical shape of the Earth is to the merits of the Greeks. Pythagoras and Kleomedes derived a spherical Earth figure from the observation of circumpolar stars. Afterwards, Eratosthenes used shadow measurements to establish north-south distances from which he deduced for the first time the approximate radius of the Earth. Beyond it, the task of geodesy was reduced to sharpen this picture for a long time. However, with the advent of modern geodesy it was not only left to the geodesists but also to the physicists to develop new procedures for the determination of the figure of the Earth. It was due to the wide range of new theories that doubts about the spherical shape of the Earth arose. Eventually two expeditions were started to the northern polar circle and to the equator. The aim was to answer the question whether the Earth is a prolate or oblate ellipsoid and what are its dimensions. The familiar result of an oblate ellipsoid confirmed Newton's mechanics. Consequently, the physical and mathematical sciences became more and more important for modern geodesy. Overall, this led to a reorganization and segmentation of geodesy into separate branches. For example the emergence of physical geodesy was accompanied by the recovery of the gravitational law and the development of potential theory. In this way, scientists became aware of the significance of the knowledge of the Earth's gravity field for the geometrical shape of the Earth's body. This intuition must be seen as the date of birth of the *geodetic boundary value problem* (GBVP), i.e. the problem of the combined determination of the Earth's figure and gravity field.

According to H. Moritz, see [66] Moritz 1980, from a theoretical point of view the GBVP represents an especially interesting and significant problem, whose importance for the conceptual structure of geodesy, from the time of Clairaut to the present day, can hardly be overestimated. In fact, the consecutive stages in the development of the GBVP – Clairaut, Stokes, Molodensky – always served as measures of perfection for geodetic theory and set new standards. The well-known formula of A.C. Clairaut from 1743, e.g. [28] Heiskanen&Moritz 1967, provided for the first time a relation between gravity and the geometric form of a level surface. In 1849, G.G. Stokes presented with his theory the first genuine solution of the GBVP. In a manner of speaking, the work of M.S. Molodensky from 1962 eventually completed the classical research on approaching the GBVP. In a sense, A.C. Clairaut was a predecessor of G.G. Stokes and M.S. Molodensky.

It can be claimed that until the advent of space techniques with the launch of the satellite Sputnik 1 in 1957, the theory was always far ahead of the data available at the time. So far, the solution of the GBVP according to Stokes and Molodensky relied on the use of terrestrial data. However, based on the sparsely available data it was simply impossible to determine the geometry of the Earth's surface as a whole or the gravity field of the Earth with high enough precision. This situation soon changed with now rapidly upcoming new satellite missions. Special missions, such as Echo 1, Explorer 19, Geos 1 or Lageos 1, provided data sets that were essentially dense enough, almost globally distributed and to a certain extent continuous. Thus, they allowed for the first time to produce reliable solutions for the gravity field of the Earth and the geoid as the physical equivalent of geometrical Earth.

In addition, the start of the global positioning system (GPS) led to the definition of a new category of boundary value problems (BVPs). Whereas the classical GBVP represents a so-called *free* boundary value problem (BVP), since the boundary surface itself, i.e. the Earth's surface or the geoid, needs to be determined as well, the new BVP category is based on a *fixed* type of problem. That means, the surface of the Earth is considered to be known from GPS measurements. Moreover, dedicated geoscientific satellite missions in the recent years created a real wealth of new BVPs. Among others, *altimetry* and *gradiometry* type of BVPs as well as *mixed* and *overdetermined* BVPs nowadays furnish the geodetic community with a detailed knowledge of the Earth's gravity field and topography.

As a result of these new techniques and the wealth of suddenly available information, the previous statement, that theory is always ahead of the available data, is probably no longer true. Hence, every once in a while a return to basic research seems advisable in order to ensure that also from a theoretical point of view the highest quality is achieved for geodetic results.

1-2 Geopotential recovery in the context of gravity space

As was mentioned above, M.S. Molodensky proposed in the early 1960s his theory to solve the GBVP. He approached the original nonlinear free problem by means of introducing suitable approximates for the Earth's topography and for Earth's gravity potential and by linearizing the problem in terms of these approximates. In the first place, adopting a reference surface, the so-called *telluroid*, for the Earth's topography changed the originally free problem into a fixed problem. In addition, the application of a normal potential simplified the initial nonlinear problem into a linear type of BVP. It is easily understandable that this twofold linearization procedure, which is also applied similarly within the scope of Stokes' theory, is quite demanding.

On this account, F. Sansò pioneered almost 30 years ago a completely new concept for solving the GBVP. His method, which has become known as the *gravity space approach*, solves the GBVP by mapping it into a dual space. Interestingly enough, this so-called *gravity space* is built upon quite specific coordinates. Since for the treatment of the GBVP the gravity vector is assumed to be known throughout the Earth's surface, its three Cartesian components are chosen to form the new independent coordinates of gravity space. This can be accomplished by using a Legendre transformation, the simplest form of a contact transformation. The outstanding achievement of F. Sansò's approach is the fact that the resulting BVP in gravity space directly constitutes a *fixed* problem. That is, the boundary surface in gravity space is actually known, which results in direct consequence of the above stated prerequisite that the gravity vector must be known throughout the surface of the Earth to solve the GBVP. The knowledge of the boundary surface is clearly in contrast to the classical free problem and a tremendous advantage of the gravity space methodology. However, the other side of the coin is the fact that F. Sansò's approach suffers from a singularity at the origin. Moreover, F. Sansò's methodology is physically more abstract. In contrast to the classical geodetic boundary value problem where the potential is required in the Earth's exterior domain, the resulting BVP in gravity space represents an interior problem in terms of the so-called *adjoint potential*. Naturally, such a situation is somewhat unusual. In addition, instead of coordinates that exhibit metric units, gravity space coordinates are given in the dimension of accelerations. And at last, within the linearized form of the classical GBVP the disturbing potential must satisfy Laplace's equation, whereas for the corresponding problem in gravity space the adjoint disturbing potential must solve the more complex Poisson's equation.

In conclusion, neither the classical solution approaches according to G.G. Stokes and M.S. Molodensky, Chapter 3, nor the idea of F. Sansò, Chapter 4, are without theoretical and practical deficiencies. This is in fact the starting point both for the work of W. Keller, Chapter 5, and for the research presented in Chapter 6 of this study to establish a revised theory.

1-3 Background for the intended research and dissertation objectives

For this reason, the present work resumes a twenty years old idea of W. Keller and enhances it. The basic principle of W. Keller's approach is to replace F. Sansò's Legendre type of gravity space formulation with a formulation based on a more sophisticated type of contact transformation. This serves the purpose of eliminating most of the shortcomings of F. Sansò's methodology. First of all, the modified gravity space approach according to W. Keller is free of any singularities. As a result, the new approach operates in the so-called *singularity-free* or *regular* gravity space. Furthermore, after linearization with respect to a *spherical* or rather *isotropic* normal potential, the resulting BVP in regular gravity space rests upon the problem of solving Laplace's equation under a linear boundary condition. In fact, this problem can be regarded as the gravity space counterpart of the simple Molodensky's problem, since it is of the same mathematical structure. These are the two major achievements compared to F. Sansò's methodology. Consequently, all known and commonly used algorithms and procedures developed for the solution of the traditional Molodensky's problem can be used further on. From this, the following tasks and objectives result for the first part of the thesis:

- Review of the classical theory related to the GBVP,
- summary of F. Sansò's gravity space concept,
- introduction of W. Keller's regular gravity space approach, thereby
- elaborating the resemblance to the simple Molodensky's problem.
- First numerical implementation of the regular gravity space approach in order to
- assess the suitability of the method.

In addition, since it will turn out during the first numerical investigations that the initial linearization procedure, despite the particularized conceptual advantages, results in computational deficiencies, the second part of the presented work is devoted to establish a modified theory based on a different gravity space mapping scheme. In fact, remedy is sought-after in adopting an *ellipsoidal* or rather *spheroidal* normal potential as linearization point and by tayloring the gravity space mapping in such a way that the linearized problem again has the same mathematical structure as the simple Molodensky's problem. Consequently, the underlying set of tasks and objectives listed already above is extended as follows:

- Introduction of a new advanced theory based on an improved linearization point along with
- a revised numerical implementation, in order to address the fundamental questions
- whether the mathematical structure of the simple Molodensky's problem can be preserved and
- whether introducing an ellipsoidal normal potential is numerically advantageous and, finally,
- whether the GBVP can also be successfully solved in regular gravity space.

Altogether, the content of this doctoral thesis has to be considered as a fundamental research work. The aim is to prove the general feasibility of the regular gravity space approaches by means of a numerical proof of concept study. At the same time, the research closes a gap in the existing theoretical foundation of the GBVP that requires attention. All in all, the present work shall contribute to a better understanding of the familiar GBVP. A methodologically sound completion of the underlying theory is intended rather than to address a scientific issue at the forefront of geodetic research.

1-4 Thesis outline

The outline of this work is as follows. Subsequent to the present introductory chapter, a compilation of selected fundamental subjects shall provide the scientific background required throughout this work. Among other things, a brief overview of several important geopotential concepts, the relevant BVPs in potential theory, the Stokes problem and the theory of contact transformations will be given in the next chapter. Thereafter, Chapters 3 to 6 present the basic theoretical principles for solving the GBVP by means of the various aforementioned approaches. Starting from Chapter 3, the classical Molodensky's theory is recapitulated. The first gravity space concept originating from F. Sansò is reviewed in Chapter 4. In Chapter 5, W. Keller's regular gravity space approach, which exploits a simple isotropic linearization procedure, is introduced in detail. Chapter 6 deals with the revised regular gravity space approach, which is based on the more advanced spheroidal linearization principle. This completes the theoretical considerations. It is worth mentioning that the material presented in Chapters 3 to 6 is organized in an analogous manner. Apart from Chapter 3, which only operates in geometry space, the mapping relations between ordinary and gravity space as well as the fundamental properties of the particular transformation

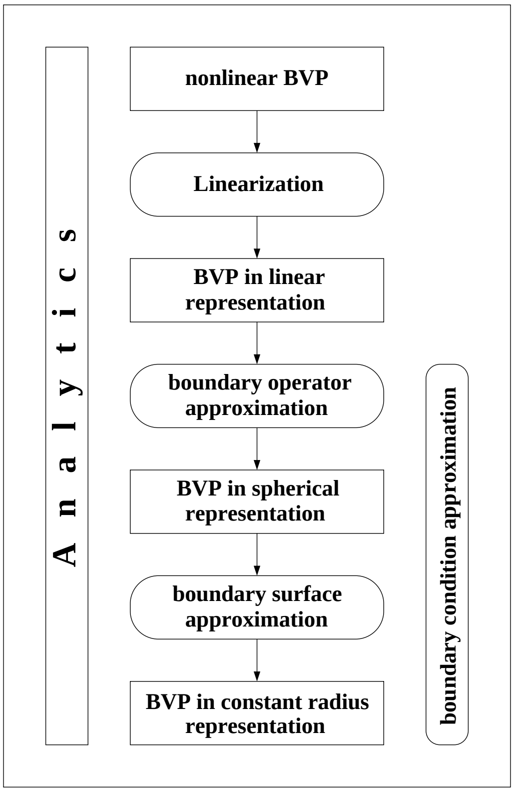


Figure 1.1: Diagram of the formal structure of Chapters 3 to 6 in line with the analytical GBVP solution procedure

are discussed at the beginning of each chapter. The further formal structure is illustrated in Fig. 1.1. If applicable, the rectangular boxes specify the individual sections of each chapter. At the same time, the figure provides an overview of the analytical procedure to achieve a representation of the GBVP finally appropriate for numerical evaluation as addressed in Chapter 7. According to Fig. 1.1, the respective nonlinear BVPs are established at first. The nonlinear BVPs differ from each other depending on the choice of the gravity space formulation. Next, the corresponding linear BVPs based on the particular linearization procedure are derived. As far as the classical solution of the GBVP is concerned, this involves, as mentioned before, the adoption of suitable approximations for the surface of the Earth and for the Earth's potential field. As far as the gravity space approaches are concerned, merely the introduction of suitable normal potential relations have to be considered. In a next step, the boundary operators associated with the linear BVPs are simplified. For each approach, the assumptions are slightly different in order to come up with the so-called BVPs in spherical representation. Subsequent to the boundary operator simplifications, the boundary surfaces themselves can be approximated as well. This implies the substitution of the actual boundary surface with a mathematically simpler surface, e.g. with a sphere. Typically, the last two modifications in common constitute the approximation of the boundary condition, which finally leads to the BVPs in constant radius approximation. Hence, after having supplied the respective BVP representations suitable for the intended computational implementation, the numerical proof of concept for both regular gravity space approaches is presented in Chapter 7. The first part has its focus on the results accomplished with W. Keller's method and the second part is devoted to the results based on the revised method with the new linearization concept. At last, Chapter 8 concludes the thesis with a final summary, a discussion of the general result and a short outlook providing several recommendations for further research.

Chapter 2

General background

This chapter is meant to provide several selected fundamental concepts, which form the basis for the further investigations. The predominantly theoretical aspects presented in the following, recur throughout this thesis and are therefore introduced in advance. In detail, the starting point is to take a glance at the two coordinate sets most commonly applied in the present work. Subsequent to this, a more general look at the subject-matter of coordinate transformations is given. The next section yields the basic relations required in dealing with the Earth's gravity field. For this purpose, three consecutive paragraphs attend to the crucial relationships in the context of the true, the normal and the anomalous geopotential field. A forth paragraph is devoted to the representation of the geopotential field in terms of spherical harmonics. Thereafter, with regard to the intended research on the GBVP, the relevant BVPs of potential theory are outlined, which leads to a classification of the BVPs in general. The focus of the subsequent section is on Stokes' problem, which represents one possibility to address the GBVP. The last section of this chapter introduces a particularly interesting family of transformations, the so-called contact transformations. It shows that these transformations establish the foundation for the gravity space formulations. Thus, they form a main issue for the current research.

2-1 Coordinate systems and related coordinate transformations

Naturally, the interrelationship of Cartesian and spherical coordinates is of utmost importance. On this account, the equations for forward and backward transformation of Cartesian and spherical coordinates are worth mentioning here.

2-1.1 Cartesian and spherical coordinates

As for an explicit definition of Cartesian or spherical coordinates the interested reader is asked to consult any standard textbook on mathematics, e.g. [8] Burg 1990. In the first place, the transformation of spherical coordinates (λ, ϕ, r) to Cartesian coordinates (x_1, x_2, x_3) is given by

$$\mathbf{x} = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = r \begin{bmatrix} \cos\lambda\cos\phi \\ \sin\lambda\cos\phi \\ \sin\phi \end{bmatrix} . \tag{2.1}$$

In reverse, the formulas to transform Cartesian coordinates (x_1, x_2, x_3) to spherical coordinates (λ, ϕ, r) are

$$\lambda = \arctan\frac{x_2}{x_1} + (-\frac{1}{2}\operatorname{sgn} x_2 - \frac{1}{2}\operatorname{sgn} x_2 \operatorname{sgn} x_1 + 1)\pi := \operatorname{atan2}\frac{x_2}{x_1} \tag{2.2}$$

$$\phi = \begin{cases} \arctan\frac{x_3}{\sqrt{x_1^2+x_2^2}} & ;\ x_1 \neq 0 \vee x_2 \neq 0 \\ \operatorname{sgn} x_3 \frac{\pi}{2} & ;\ (x_1 = 0 \wedge x_2 = 0) \wedge x_3 \neq 0 \end{cases} \tag{2.3}$$

$$r = \|\mathbf{x}\| = +\sqrt{x_1^2 + x_2^2 + x_3^2}\,, \tag{2.4}$$

subject to

$$\operatorname{sgn} x = \begin{cases} -1 & ; \quad \text{for } x < 0 \\ 0 & ; \quad \text{for } x = 0 \\ 1 & ; \quad \text{for } x > 0\,. \end{cases} \tag{2.5}$$

Moreover, for some applications the following relations are of particular interest

$$\sin\lambda = \frac{x_2}{\sqrt{x_1^2+x_2^2}} \quad ; \quad \cos\lambda = \frac{x_1}{\sqrt{x_1^2+x_2^2}} \tag{2.6}$$

$$\sin\phi = \frac{x_3}{r} \quad ; \quad \cos\phi = \frac{\sqrt{x_1^2+x_2^2}}{r}\,. \tag{2.7}$$

2-1.2 Jacobian matrix and Jacobian determinant

A set of rectangular Cartesian coordinates, expressed by

$$\mathbf{x} = \begin{bmatrix} x_1 & x_2 & x_3 \end{bmatrix}^\top , \tag{2.8}$$

is related to a second set of coordinates

$$\mathbf{p} = \begin{bmatrix} p_1 & p_2 & p_3 \end{bmatrix}^\top , \tag{2.9}$$

such as e.g. the curvilinear triple (λ, ϕ, r) as introduced in the last paragraph, by equations of the following form

$$p_1 = p_1(x_1, x_2, x_3) \quad ; \quad p_2 = p_2(x_1, x_2, x_3) \quad ; \quad p_3 = p_3(x_1, x_2, x_3)\ , \tag{2.10}$$

see (2.2), (2.3) and (2.4), or in a more compact notation

$$p_i = p_i(x_j) \qquad i,j = 1,2,3\,. \tag{2.11}$$

The Jacobian matrix $\mathbf{J}$, i.e. the matrix of first-order partial derivatives, related to the coordinate transformation (2.11) is explicitly given as

$$\mathbf{J} = \left[\frac{\partial p_i}{\partial x_j}\right] = \begin{bmatrix} \frac{\partial p_1}{\partial x_1} & \frac{\partial p_1}{\partial x_2} & \frac{\partial p_1}{\partial x_3} \\ \frac{\partial p_2}{\partial x_1} & \frac{\partial p_2}{\partial x_2} & \frac{\partial p_2}{\partial x_3} \\ \frac{\partial p_3}{\partial x_1} & \frac{\partial p_3}{\partial x_2} & \frac{\partial p_3}{\partial x_3} \end{bmatrix} \tag{2.12}$$

and since (2.11) is assumed to represent an uniquely defined and invertible transformation, the condition

$$|\mathbf{J}| = \left|\left[\frac{\partial p_i}{\partial x_j}\right]\right| \neq 0 \qquad i,j = 1,2,3 \tag{2.13}$$

holds. Thus, since the determinant of the Jacobian matrix $|\mathbf{J}|$ is unequal to zero, the transformation

$$x_j = x_j(p_k) \qquad j,k = 1,2,3 \tag{2.14}$$

exists, cf. (2.1). Analogously, the Jacobian matrix $\bar{\mathbf{J}}$ of the inverse transformation (2.14) is

$$\bar{\mathbf{J}} = \left[\frac{\partial x_j}{\partial p_k}\right] = \begin{bmatrix} \frac{\partial x_1}{\partial p_1} & \frac{\partial x_1}{\partial p_2} & \frac{\partial x_1}{\partial p_3} \\ \frac{\partial x_2}{\partial p_1} & \frac{\partial x_2}{\partial p_2} & \frac{\partial x_2}{\partial p_3} \\ \frac{\partial x_3}{\partial p_1} & \frac{\partial x_3}{\partial p_2} & \frac{\partial x_3}{\partial p_3} \end{bmatrix} \tag{2.15}$$

and the corresponding determinant of the Jacobian matrix $|\bar{\mathbf{J}}|$ is also unequal to zero

$$|\bar{\mathbf{J}}| = \left|\left[\frac{\partial x_j}{\partial p_k}\right]\right| \neq 0 \qquad j,k = 1,2,3\,. \tag{2.16}$$

Furthermore, the following statement for the relationship of the Jacobian matrices $\mathbf{J}$ and $\bar{\mathbf{J}}$ can be made:

Lemma 1 *The Jacobi matrix $\mathbf{J}$ of the coordinate transformation in (2.11) is related to the Jacobi Matrix $\bar{\mathbf{J}}$ of the corresponding inverse coordinate transformation (2.14) by*

$$\bar{\mathbf{J}} = \mathbf{J}^{-1}\,.$$

Proof.

$$\mathbf{J}\bar{\mathbf{J}} = \left[\frac{\partial p_i}{\partial x_j}\right]\left[\frac{\partial x_j}{\partial p_k}\right] = \left[\frac{\partial p_i}{\partial x_j}\frac{\partial x_j}{\partial p_k}\right] = \left[\frac{\partial p_i}{\partial p_k}\right] = [\delta_{ik}] = \mathbf{I} \tag{2.17}$$

Hence, it follows directly

$$\bar{\mathbf{J}} = \mathbf{J}^{-1}\,. \qquad \diamondsuit$$

In other words, it follows from Lemma 1 that the inverse of the Jacobian matrix $\mathbf{J}$ is simply the Jacobian matrix of the inverse transformation – a relation, which will be of importance several times in the context of this work.

2-2 Geopotential concepts

This section deals with the basic principles and definitions needed in the discipline of gravity field research. In the first place, all involved quantities and deduced functionals that are required throughout, will be introduced and their physical properties pointed out. In detail, the first paragraph discusses briefly the difference of the true gravity potential and the true gravitational potential of the Earth as well as the subject of actual Earth's gravity and gravitation. In a similar manner, the corresponding normal quantities are recalled in the second paragraph. Consistently, the third paragraph summarizes the resulting anomalous quantities. At last, the task of modeling the geopotential and its deduced functionals in terms of spherical harmonics series expansions is investigated.

To begin with, some agreements on the formal character of the underlying notation shall be reached. In the sequel, following the example given in [50] Kellogg 1967, use of the symbolic vector

$$\nabla = \begin{bmatrix} \frac{\partial}{\partial x_1} & \frac{\partial}{\partial x_2} & \frac{\partial}{\partial x_3} \end{bmatrix}^\top \tag{2.18}$$

is made, which is given in the coordinates (x_1, x_2, x_3) of a three-dimensional Cartesian coordinate system and represented by the so-called *nabla* symbol ∇. Eq. (2.18) is a vector differential operator, meaningless when standing alone, but frequently used in vector calculus to define quantities such as the gradient of a scalar function $\phi(\mathbf{x}) : \mathbb{R}^3 \to \mathbb{R}^1$

$$\operatorname{grad} \phi = \nabla \phi \tag{2.19}$$

or divergence and curl of a vector field $\mathbf{f}(\mathbf{x}) : \mathbb{R}^3 \to \mathbb{R}^3$ and $\mathbf{f}(\mathbf{x}) = \begin{bmatrix} f_1(\mathbf{x}) & f_2(\mathbf{x}) & f_3(\mathbf{x}) \end{bmatrix}^\top$

$$\operatorname{div} \mathbf{f} = \nabla \cdot \mathbf{f} = \frac{\partial f_1}{\partial x_1} + \frac{\partial f_2}{\partial x_2} + \frac{\partial f_3}{\partial x_3} \tag{2.20}$$

$$\operatorname{curl} \mathbf{f} = \nabla \times \mathbf{f} = \begin{bmatrix} \frac{\partial f_3}{\partial x_2} - \frac{\partial f_2}{\partial x_3} & \frac{\partial f_1}{\partial x_3} - \frac{\partial f_3}{\partial x_1} & \frac{\partial f_2}{\partial x_1} - \frac{\partial f_1}{\partial x_2} \end{bmatrix}^\top , \tag{2.21}$$

as will be done throughout this work. Furthermore, the Laplacian Δ (or ∇^2) of a scalar field $\phi(\mathbf{x}) : \mathbb{R}^3 \to \mathbb{R}^1$ is the divergence of the field's gradient $\nabla\phi$, i.e.

$$\Delta\phi = \nabla \cdot \nabla\phi = \frac{\partial^2 \phi}{\partial x_1^2} + \frac{\partial^2 \phi}{\partial x_2^2} + \frac{\partial^2 \phi}{\partial x_3^2} . \tag{2.22}$$

2-2.1 True geopotential fields

As mentioned before, the short revision of the fundamental geopotential concepts begins with a review on the actual field parameters. As is known, the true gravity potential of the Earth W is composed of the true terrestrial gravitational potential V, which is according to Newton generated by the integrated attraction of mass elements on a test particle, and the centrifugal potential Z

$$W(\mathbf{x}) = V(\mathbf{x}) + Z(\mathbf{x}) . \tag{2.23}$$

The gravity vector $\mathbf{\Gamma}(\mathbf{x})$ is related to the gravity potential $W(\mathbf{x})$ by taking the gradient

$$\mathbf{\Gamma}(\mathbf{x}) = \nabla W(\mathbf{x}) = \left[\frac{\partial W}{\partial x_i} \right] . \tag{2.24}$$

Comparable with (2.1), the vector $\mathbf{\Gamma} = [\Gamma_i]$ can be expressed in terms of measured gravity $\Gamma = \|\mathbf{\Gamma}\|$, astronomical latitude Φ and longitude Λ as follows

$$\mathbf{\Gamma} = \nabla W = \begin{bmatrix} \Gamma_1 \\ \Gamma_2 \\ \Gamma_3 \end{bmatrix} = -\Gamma \begin{bmatrix} \cos\Phi\cos\Lambda \\ \cos\Phi\sin\Lambda \\ \sin\Phi \end{bmatrix} . \tag{2.25}$$

Accordingly, the Hessian matrix $\mathbf{M}$ is composed of the second-order partial derivatives of the gravity potential W

$$\mathbf{M} = [M_{ij}] = [\nabla(\nabla W)] = \left[\frac{\partial W}{\partial x_i \partial x_j} \right] = \nabla\mathbf{\Gamma} = \left[\frac{\partial \Gamma_i}{\partial x_j} \right] . \tag{2.26}$$

Similarly to (2.24), application of the gradient operator relates the gravitational potential $V(\mathbf{x})$ to the gravitational acceleration vector $\mathbf{g}(\mathbf{x})$. Consequently,

$$\mathbf{g} = [g_i] = \nabla V = \left[\frac{\partial V}{\partial x_i}\right] \tag{2.27}$$

holds true. Furthermore, the second-order partial derivatives of the gravitational potential V form again the corresponding Hessian matrix denoted by $\mathbf{V}$. Hence, it applies

$$\mathbf{V} = [V_{ij}] = [\nabla(\nabla V)] = \left[\frac{\partial V}{\partial x_i \partial x_j}\right] = \nabla \mathbf{g} = \left[\frac{\partial g_i}{\partial x_j}\right] . \tag{2.28}$$

As far as the two potential functions W and V are concerned, two relationships of fundamental significance must be emphasized. That is, in the mass-free exterior of the Earth, the gravity potential W satisfies Poisson's equation

$$\Delta W = \nabla \cdot \nabla W = 2\omega^2 \neq 0 \, , \tag{2.29}$$

whereas the gravitational potential V satisfies Laplace's equation

$$\Delta V = \nabla \cdot \nabla V = 0 \, , \tag{2.30}$$

which is, instead of the former inhomogeneous relationship, a homogeneous linear partial differential equation of second order. Since a function is by definition called harmonic in a region if it satisfies Laplace's equation at every point of this region, it follows that in contrast to W only V represents a harmonic function. Moreover, if the region is the exterior of a certain closed surface, e.g. the exterior of the Earth's surface, then the potential V must in addition vanish like $1/r$ and is regular at infinity. As a result, it can be shown that every harmonic function is analytic in the region where it satisfies Laplace's equation, i.e. it is continuous and has continuous derivatives of any order, [28] HEISKANEN&MORITZ 1967.

At last, the following two relations are worth mentioning here. In the first place, (2.30) can also be written in the form given below by taking (2.22) and (2.28) into account

$$\Delta V = \mathrm{tr}\, \mathbf{V} = 0 \, . \tag{2.31}$$

Secondly, the condition

$$\nabla \times \mathbf{\Gamma} = \nabla \times \nabla W = \mathbf{0} \tag{2.32}$$

manifests an important property of the gravity field. Namely, that $\mathbf{\Gamma}$ is *irrotational*, which follows of course from the fact that W represents a potential function and, accordingly, that the gravity field of the Earth represents a conservative force field.

2-2.2 Normal geopotential fields

Similarly to the actual gravity potential W in (2.23), the normal gravity potential W_0 is composed of the normal gravitational potential V_0 and the normal centrifugal potential Z_0

$$W_0(\mathbf{x}) = V_0(\mathbf{x}) + Z_0(\mathbf{x}) \, . \tag{2.33}$$

Usually, the normal centrifugal potential Z_0 is defined in such a way that it equals the true normal potential Z, i.e.

$$Z_0(\mathbf{x}) = Z(\mathbf{x}) \, . \tag{2.34}$$

Furthermore, the normal gravity vector $\mathbf{\Gamma}_0(\mathbf{x})$ is deduced from the normal gravity potential $W_0(\mathbf{x})$ in the same way as the gravity vector is computed from the gravity potential, that is by applying the gradient operator

$$\mathbf{\Gamma}_0(\mathbf{x}) = \nabla W_0(\mathbf{x}) \, . \tag{2.35}$$

In the same way as the gravity vector has been expressed by the coordinates (Γ, Φ, Λ), cf. (2.25), the normal gravity vector $\mathbf{\Gamma}_0 = \left[\Gamma_i^0\right]$ can be given in terms of normal gravity $\Gamma_0 = \|\mathbf{\Gamma}_0\|$, normal latitude $\hat{\phi}$ and normal longitude $\hat{\lambda}$

$$\mathbf{\Gamma}_0 = \nabla W_0 = \begin{bmatrix} \Gamma_1^0 \\ \Gamma_2^0 \\ \Gamma_3^0 \end{bmatrix} = -\Gamma_0 \begin{bmatrix} \cos\hat{\phi}\cos\hat{\lambda} \\ \cos\hat{\phi}\sin\hat{\lambda} \\ \sin\hat{\phi} \end{bmatrix} . \tag{2.36}$$

In an analogous manner to (2.26), the matrix $\mathbf{M}_0$ is composed of the second-order partial derivatives of the normal gravity potential W_0

$$\mathbf{M}_0 = \left[M_{ij}^0\right] = [\nabla(\nabla W_0)] = \left[\frac{\partial W_0}{\partial x_i \partial x_j}\right] = \nabla \mathbf{\Gamma}_0 = \left[\frac{\partial \Gamma_i^0}{\partial x_j}\right] . \tag{2.37}$$

Note, in particular in connection with the Hessian matrix $\mathbf{M}_0$, but also for the Hessian matrix $\mathbf{M}$, see (2.26), the terms *Eötvös* or *Marussi* tensor are also commonly used to refer to the matrices of second-order partial derivatives. In this special context, the matrix elements M_{ij}^0, and M_{ij} respectively, are denoted by *gravity gradients*. In many cases, the naming Eötvös or Marussi tensor is also applied to the gravitational counterparts.

Analogously to (2.35), the normal gravitational acceleration vector $\mathbf{g}_0(\mathbf{x})$ is obtained from the normal gravitational potential $V_0(\mathbf{x})$ by taking the gradient, that is

$$\mathbf{g}_0 = \left[g_i^0\right] = \nabla V_0 = \left[\frac{\partial V_0}{\partial x_i}\right] . \tag{2.38}$$

In addition, the second-order partial derivatives of the normal gravitational potential V_0 constitute the Hessian matrix $\mathbf{V}_0$ as follows

$$\mathbf{V}_0 = \left[V_{ij}^0\right] = [\nabla(\nabla V_0)] = \left[\frac{\partial V_0}{\partial x_i \partial x_j}\right] = \nabla \mathbf{g}_0 = \left[\frac{\partial g_i^0}{\partial x_j}\right] . \tag{2.39}$$

Usually, the potential of simple bodies, such as the sphere or the ellipsoid of revolution with specifically structured density distribution, is used to define a reference potential V_0. E.g. the gravitational potential of a sphere with homogeneous or radially symmetric mass distribution reads as

$$V_0(\mathbf{x}) = \frac{GM}{\|\mathbf{x}\|} . \tag{2.40}$$

Due to the evident symmetry properties, the above normal potential represents an *isotropic* normal field, the simplest imaginable possibility. The corresponding normal gravitational acceleration vector is found by

$$\mathbf{g}_0 = \nabla V_0 = \begin{bmatrix} g_1^0 \\ g_2^0 \\ g_3^0 \end{bmatrix} = -\frac{GM}{\|\mathbf{x}\|^3}\mathbf{x} = -\frac{GM}{\|\mathbf{x}\|^2} \begin{bmatrix} \cos\phi\cos\lambda \\ \cos\phi\sin\lambda \\ \sin\phi \end{bmatrix} . \tag{2.41}$$

As for the use of a higher order reference potential, such as the potential of a level ellipsoid, it must for now be referred to Chapter 6 and standard literature, e.g. [66] Moritz 1980.

2-2.3 Anomalous geopotential fields

The disturbing gravity potential $\delta W(\mathbf{x})$ is taken as the difference of the actual gravity potential $W(\mathbf{x})$ and the normal gravity potential $W_0(\mathbf{x})$

$$\delta W(\mathbf{x}) = W(\mathbf{x}) - W_0(\mathbf{x}) . \tag{2.42}$$

Analogously, the disturbing gravitational potential $\delta V(\mathbf{x})$ is taken as the difference of the actual gravitational potential $V(\mathbf{x})$ and the normal gravitational potential $V_0(\mathbf{x})$

$$\delta V(\mathbf{x}) = V(\mathbf{x}) - V_0(\mathbf{x}) . \tag{2.43}$$

In fact, since throughout this work the normal centrifugal potential $Z_0(\mathbf{x})$ is considered to be identical to the actual centrifugal potential $Z(\mathbf{x})$, the following relation is true

$$\delta W(\mathbf{x}) = \delta V(\mathbf{x}) . \tag{2.44}$$

Hence, due to $Z_0(\mathbf{x}) = Z(\mathbf{x})$ the centrifugal potential terms involved on the right-hand side of (2.42) cancel out and the disturbing gravity potential equals the disturbing gravitational potential. Thus, in contrast to the gravity potential $W(\mathbf{x})$, which satisfies Poisson's equation, cf. (2.29), the disturbing gravity potential $\delta W(\mathbf{x})$ satisfies Laplace's equation in the mass-free domain outside the boundary surface

$$\Delta \delta W(\mathbf{x}) = 0 . \tag{2.45}$$

Within the framework of the afore given definitions, none of the quantities introduced so far has been associated with any specific boundary or reference surface. In the following, several quantities are to be considered, where restrictions to certain surfaces apply. For example, the difference of the actual potential at the Earth's surface σ and the reference potential at the reference surface Σ is denoted by *gravity potential anomaly*

$$\Delta w = W\big|_{\sigma} - W_0\big|_{\Sigma}\,, \tag{2.46}$$

whereas the *gravity potential disturbance* is, in contrast to (2.46), the difference of the true and the reference potential at the same point, e.g. at the Earth's surface

$$\delta w = \delta W\big|_{\sigma} = W\big|_{\sigma} - W_0\big|_{\sigma}\,. \tag{2.47}$$

In accordance to (2.46), the difference of the true gravity vector and the normal gravity vector, where the former is given at the Earth's surface σ and the latter at the reference surface Σ, defines the *gravity anomaly vector*

$$\Delta\mathbf{\Gamma} = \mathbf{\Gamma}\big|_{\sigma} - \mathbf{\Gamma}_0\big|_{\Sigma}\,. \tag{2.48}$$

Consequently, if in view of (2.47) the difference is taken at the same point, e.g. at the Earth's surface, the *gravity disturbance vector* is obtained

$$\delta\tilde{\mathbf{\Gamma}} = \delta\mathbf{\Gamma}\big|_{\sigma} = \mathbf{\Gamma}\big|_{\sigma} - \mathbf{\Gamma}_0\big|_{\sigma}\,. \tag{2.49}$$

Moreover, the corresponding scalar quantities are referred to as *gravity anomaly*

$$\Delta\Gamma = \Gamma\big|_{\sigma} - \Gamma_0\big|_{\Sigma} \tag{2.50}$$

and, respectively, as *gravity disturbance*

$$\delta\Gamma = \Gamma\big|_{\sigma} - \Gamma_0\big|_{\sigma}\,. \tag{2.51}$$

Finally, the following universal relationship between the general gravity disturbance vector $\delta\mathbf{\Gamma}$ and the general disturbing gravity potential δW is of importance for later considerations

$$\delta\mathbf{\Gamma} = (\mathbf{\Gamma} - \mathbf{\Gamma}_0) = (\nabla W - \nabla W_0) = \nabla\,(W - W_0) = \nabla\delta W\,. \tag{2.52}$$

Thus, the gravity disturbance vector $\delta\mathbf{\Gamma}$ equals the gradient of the disturbing gravity potential $\nabla\delta W$. Hence, $\delta\mathbf{\Gamma}$ represents the first order functional of the disturbing gravity potential δW.

2-2.4 Representing the terrestrial gravitational potential

In Section 2-2.1, the geopotential V has been found to satisfy Laplace's equation, cf. (2.30) or (2.31), and consequently to hold the property of harmonicity outside the masses of the Earth. Probably the most prominent representative of the family of harmonic functions are the so-called *spherical harmonics*, see e.g. [30] HOBSON 1931, which serve as orthogonal base functions on the unit sphere. Furthermore, spherical harmonics, as well as a series expansion of spherical harmonics, represent a solution of Laplace's equation. On this account, they are well suited to characterize a model for the potential V. Thus, the usual representation of the Earth's gravitational potential V in terms of an infinite spherical harmonic series expansion reads as follows

$$V(\lambda,\phi,r) = \frac{GM}{r}\lim_{K\to\infty}\sum_{k=0}^{K}\sum_{l=0}^{k}\left(\frac{R}{r}\right)^{k} P^*_{kl}(\sin\phi)\left[c^*_{kl}\cos(l\lambda) + s^*_{kl}\sin(l\lambda)\right]\,. \tag{2.53}$$

In (2.53), c^*_{kl}, s^*_{kl} denote the normalized spherical harmonic coefficients, which are also commonly addressed by *Stokes coefficients*, and $P^*_{kl}(\sin\phi)$ are referred to as Ferrer's fully normalized associated Legendre functions of degree k and order l. The aspect of deriving the functions $P^*_{kl}(\sin\phi)$ via recurrence formulae is treated in Appendix A. As before, (λ,ϕ,r) are spherical coordinates, GM denotes the product of gravitational constant G and the Earth's mass M and R refers to the equatorial radius of the Earth, which is a specific constant for each individual geopotential model. However, instead of utilizing the infinite series representation according to (2.53), the gravitational potential V is approximated in terms of the following limited series expansion

$$V(\lambda,\phi,r) = \frac{GM}{r}\left(1 + \sum_{k=2}^{K}\sum_{l=0}^{k}\left(\frac{R}{r}\right)^{k} P^*_{kl}(\sin\phi)[c^*_{kl}\cos(l\lambda) + s^*_{kl}\sin(l\lambda)]\right)\,. \tag{2.54}$$

In (2.54), the parameter K has to be understood as maximum degree K_{max}, which reflects the degree of resolution of the spherical harmonics series expansion. The value K_{max} is thereby dependent on a variety of factors related to the underlying data used to determine the Stokes coefficients. In this context criteria such as spatial data density, data sensitivity, data quality or the involved measurement principle are noteworthy. Moreover, representing the gravitational potential of the Earth in the form given by (2.54) directly implies $c^*_{00} = 1$ and $c^*_{10} = c^*_{11} = s^*_{11} = 0$, which accounts for the fact that the origin of the underlying coordinate system is assumed to coincide with the center of mass of the Earth. At last, the forward computation should be commented on. That is, in view of (2.54), the determination of potential values from known Stokes coefficients, a procedure which is usually referred to as *spherical harmonic synthesis*.

For reasons of compactness, two alternative representations of (2.54) are widely used. On the one hand the potential V can be expanded into a series of spherical harmonics by making use of Laplace's *surface spherical harmonics* $\bar{Y}_{kl}(\lambda, \phi)$. In that case, for the potential V holds the following expression

$$V(\lambda, \phi, r) = \frac{GM}{R}\left(1 + \sum_{k=2}^{K}\sum_{l=-k}^{k} \bar{c}_{kl}\left(\frac{R}{r}\right)^{k+1} \bar{Y}_{kl}(\lambda, \phi)\right), \tag{2.55}$$

which is based on a combination of the Stokes coefficients according to

$$\bar{c}_{kl} := \begin{cases} c^*_{kl} & ; \quad 0 \le l \le k \\ s^*_{k|l|} & ; \ -k \le l < 0 \end{cases} \tag{2.56}$$

and the product of the *attenuation factor* $\left(\frac{R}{r}\right)^{k+1}$ and the surface spherical harmonics $\bar{Y}_{kl}(\lambda, \phi)$. The functions $\bar{Y}_{kl}(\lambda, \phi)$ are specified as follows

$$\bar{Y}_{kl}(\lambda, \phi) := \begin{cases} P^*_{kl}(\sin\phi)\cos(l\lambda) & ; \quad 0 \le l \le k \\ P^*_{k|l|}(\sin\phi)\sin(|l|\,\lambda) & ; \ -k \le l < 0\,. \end{cases} \tag{2.57}$$

On the other hand, the potential V can be developed into a series of spherical harmonics by making direct use of the so-called *solid spherical harmonics* $\bar{\Upsilon}_{kl}(\lambda, \phi, r)$. That is, V is given by

$$V(\lambda, \phi, r) = \frac{GM}{R}\left(1 + \sum_{k=2}^{K}\sum_{l=-k}^{k} \bar{c}_{kl}\,\bar{\Upsilon}_{kl}(\lambda, \phi, r)\right), \tag{2.58}$$

with the relationship

$$\bar{\Upsilon}_{kl}(\lambda, \phi, r) := \left(\frac{R}{r}\right)^{k+1} \bar{Y}_{kl}(\lambda, \phi) \tag{2.59}$$

explicitly defining the solid spherical harmonics in terms of the previously mentioned product of the attenuation term and the surface spherical harmonics. Note that only the *solid* spherical harmonics $\bar{\Upsilon}_{kl}(\lambda, \phi, r)$ constitute harmonic functions. Consequently, every harmonic function can be expressed as a product-sum in terms of solid spherical harmonics.

Remark 1 Most notable is the importance of the following identity

$$\int_{-\frac{\pi}{2}}^{+\frac{\pi}{2}}\int_{0}^{2\pi} \bar{Y}_{kl}(\lambda, \phi)\bar{Y}_{mn}(\lambda, \phi)\cos\phi \,\mathrm{d}\lambda\mathrm{d}\phi = \begin{cases} 4\pi & ; \ (k = m) \wedge (l = n) \\ 0 & ; \ otherwise \end{cases}, \tag{2.60}$$

which constitutes the orthonormality relation of the surface spherical harmonics in case of a unit sphere. Eq. (2.60) becomes important for the realization of a *spherical harmonic analysis*, a procedure for the determination of Stokes coefficients from data, e.g. potential values, given all-over the sphere. See Appendix C-2 for more details.

At last, it will prove useful to provide the representation of the gravitational acceleration vector $\mathbf{g}$ in spherical coordinates (λ, ϕ, r). Hence, comparable to(2.27), i.e.

$$\mathbf{g}_{\lambda,\phi,r}(\lambda, \phi, r) = \nabla_{\lambda,\phi,r} V(\lambda, \phi, r)\,, \tag{2.61}$$

the *nabla* operator, see (2.18), in spherical coordinates

$$\nabla_{\lambda,\phi,r} = \left[\begin{array}{ccc} \frac{1}{r\cos\phi}\frac{\partial}{\partial\lambda} & \frac{1}{r}\frac{\partial}{\partial\phi} & \frac{\partial}{\partial r} \end{array} \right]^{\top} \tag{2.62}$$

has to be specified at first. Then (2.61) explicitly reads as follows

$$\mathbf{g}_{\lambda,\phi,r}(\lambda,\phi,r) = \nabla_{\lambda,\phi,r} V(\lambda,\phi,r) = \left[\begin{array}{ccc} \frac{V_\lambda(\lambda,\phi,r)}{r\cos\phi} & \frac{V_\phi(\lambda,\phi,r)}{r} & V_r(\lambda,\phi,r) \end{array} \right]^{\top}, \tag{2.63}$$

subject to the first order partial derivatives of the potential V

$$V_\lambda(\lambda,\phi,r) = \frac{\partial V(\lambda,\phi,r)}{\partial\lambda} = \frac{GM}{r}\sum_{k=2}^{K}\sum_{l=0}^{k}\left(\frac{R}{r}\right)^k P^*_{kl}(\sin\phi)[-c^*_{kl} l\sin(l\lambda) + s^*_{kl} l\cos(l\lambda)] \tag{2.64}$$

$$V_\phi(\lambda,\phi,r) = \frac{\partial V(\lambda,\phi,r)}{\partial\phi} = \frac{GM}{r}\sum_{k=2}^{K}\sum_{l=0}^{k}\left(\frac{R}{r}\right)^k P^{*\prime}_{kl}(\sin\phi)[c^*_{kl}\cos(l\lambda) + s^*_{kl}\sin(l\lambda)] \tag{2.65}$$

$$V_r(\lambda,\phi,r) = \frac{\partial V(\lambda,\phi,r)}{\partial r} = -\frac{GM}{r^2}\left(1 + \sum_{k=2}^{K}\sum_{l=0}^{k}(k+1)\left(\frac{R}{r}\right)^k P^*_{kl}(\sin\phi)[c^*_{kl}\cos(l\lambda) + s^*_{kl}\sin(l\lambda)]\right), \tag{2.66}$$

and, in (2.65),

$$P^{*\prime}_{kl}(\sin\phi) = \frac{dP^*_{kl}(\sin\phi)}{d\phi} \tag{2.67}$$

denotes the derivative of the Legendre function $P^*_{kl}(\sin\phi)$ with respect to ϕ. Formulae for the recursive computation of the first order functionals $P^{*\prime}_{kl}(\sin\phi)$ are compiled in Appendix A.

Remark 2 On the basis of the above relations, establishing the gradient of the gravitational potential V, the direction of the greatest rate of increase of V can be identified. Hence, a well-known property of the gravitational potential V, i.e. being predominantly radially dependent, can be directly understood and to some extent quantified. In fact, a comparison of the magnitudes of the components g_i of the gravitational acceleration vector $\mathbf{g}$, when represented in spherical coordinates, reveals that the last component of (2.63), based on the radial derivative (2.66), is larger by a factor of roughly 10^3 than the first two components of (2.63), based on the derivatives (2.64) and (2.65).

2-3 Boundary value problems in potential theory

In general, a BVP is a differential equation together with a set of additional constraints, called the boundary conditions. A solution to the BVP is a solution to the differential equation which also satisfies the boundary conditions. A boundary condition represents a set of values assigned on the physical boundary of the solution domain, i.e. the region in which the problem is specified. Two cases are usually considered. First, within the scope of a so-called *internal* BVP the boundary curve, respectively the boundary surface, encloses the solution domain. As for the so-called *external* BVP, the situation is reversed. The boundary surface is bounded by the region where the differential equation applies. In the present context of gravity field determination predominantly external BVPs are treated. That is, the solution domain of Laplace's partial differential equation corresponds to the space outside the Earth's surface σ, which is considered as a closed, sufficiently smooth surface.

2-3.1 General remarks

Naturally, the first question that arises in this respect is whether a function V harmonic outside a surface σ is uniquely determined by its values on σ provided that σ is a closed and sufficiently smooth surface. The answer provides *Stokes' theorem*, which states that there is only one harmonic function V that assumes given boundary values on the surface σ, cf. e.g. [28] Heiskanen&Moritz 1967. Of course this directly leads to the second question, which attends to the problem whether such a harmonic function always exists. The clue to the puzzle is called *Dirichlet's principle*, which constitutes the assertion that for arbitrarily prescribed boundary values there always exists a harmonic function V that assumes on σ the given boundary values, see again [28] Heiskanen&Moritz 1967. Hence, that sets the stage for the discussion of the different types of BVPs.

2-3.2 Problems of Dirichlet, Neumann and Robin

It is common practice to distinguish BVPs of potential theory by the type of the associated boundary condition. Among the earliest boundary value problems to be studied is *Dirichlet's problem*, which is the solution of the previously outlined problem of computing the harmonic function V outside σ from its boundary values v on σ. This BVP is addressed next. Thereafter, some other significant types of BVPs will be shortly reviewed, see e.g. [61] Martensen&Ritter 1997.

Dirichlet's problem. Dirichlet's problem or the *first BVP of potential theory* is the problem of finding the connection between a continuous function v on the boundary surface $\sigma = \partial\Omega$ of a region Ω with a harmonic function V taking on the values of the prescribed function v on σ. In case the boundary surface σ is a sphere, the first BVP of potential theory can be directly solved, i.e. the harmonic function V can be determined, either by Poisson's integral or by means of the application of spherical harmonics.

Neumann's problem. Neumann's problem or the *second BVP of potential theory* differs in so far from the previous BVP that normal derivative values $\partial V/\partial n$ are given on the surface σ instead of values v of the function V itself. Usually, the normal derivative is the derivative along the outward-directed surface normal n to σ. In case the boundary surface σ is again a sphere, the second BVP of potential theory can also be solved by means of spherical harmonics or via the use of the so-called Neumann's function, in many cases also referred to as Hotine's function.

Robin's problem. In Robin's problem, also referred to as Poincarè's problem, or the *third BVP of potential theory* a linear combination of values v of V and of its normal derivative values $\partial V/\partial n$ is given on S. Hence, this type of BVP represents a combination of the Dirichlet boundary condition and the Neumann boundary condition. In case the boundary surface σ is once more the sphere, the third BVP of potential theory can be again solved by spherical harmonics or by means of Stokes' integral.

Oblique-derivative problems. Oblique-derivative problems occur whenever the derivative of the function V has to be taken in a direction other than the direction of the surface normal. As will be seen, this is in fact the more usual case in geodesy.

In conclusion, all BVPs described above share the same requirements. The demanded function V must fulfill a certain field equation, which is usually Laplace's partial differential equation. In addition, for the determination of the function V a boundary condition needs to be satisfied. The associated boundary operator represents any combination of the considered harmonic function V itself and/or of corresponding first-order functionals of V. This implies a restriction of the boundary operator onto a certain boundary surface σ, which in turn has to meet certain mathematical requirements. Within the scope of the BVPs in potential theory this boundary surface is generally required to be known. This is in contrast to the GBVP, the central BVP of physical geodesy, where also the boundary surface must be determined. Apart from that, the problem of determining the figure of the Earth from measurements taken at the Earth's surface constitutes in its linearized form a third BVP of potential theory. Therefore the third BVP of potential theory is also referred to as BVP of physical geodesy. Moreover, the GBVP in its general form belongs to the category of oblique-derivative problems. It will be discussed in detail starting from the next chapter.

2-4 Stokes problem

According to H. Moritz, see [66] Moritz 1980, Stokes's problem is the GBVP in its simplest form. Provided that the data, i.e. gravity anomalies $\Delta\Gamma$, are given on the sphere S, the disturbing gravity potential δW must be determined on and outside S, thereby assuming δW to be harmonic outside S. That is,

$$\Delta \delta W(\mathbf{x}) = 0, \quad \mathbf{x} \in ext\ S \tag{2.68}$$

holds. The corresponding boundary condition, e.g. [28] Heiskanen&Moritz 1967, reads as

$$\left(\frac{\partial \delta W}{\partial r} + \frac{2}{R}\delta W \right)\Bigg|_S = -\Delta\Gamma\,. \tag{2.69}$$

In (2.69), the free parameter r represents the norm of $\mathbf{x}$, cf. (2.4), and the constant R denotes the radius of the mean or reference sphere. Moreover, the radial direction $\frac{\partial}{\partial r}$ is normal to the bounding sphere S. Hence, the oblique-derivative problem that is generally involved in the case when dealing with the GBVP, cf. Section 2-3.2, is reduced to the above BVP, which is subject to the normal derivative boundary condition (2.69). As a consequence thereof, the BVP established by means of (2.68), (2.69) essentially represents a third BVP of potential theory. Solving such a normal derivative problem is of course much simpler than an oblique-derivative problem. This will become clear in the further investigations presented in this work. After having determined the disturbing gravity potential δW as the solution of the BVP constituted by (2.68) and (2.69), the *geoidal height* or the *geoidal undulation* N, thus the geoid, is obtained from *Bruns' formula*

$$N = \frac{\delta W}{\Gamma_0}\,, \tag{2.70}$$

which relates the geoidal undulation N to the disturbing gravity potential δW. In good approximation, instead of applying the true normal gravity Γ_0 in (2.70), a global mean gravity value $\bar{\Gamma}$ can be substituted for Γ_0.
In short, the problem of Stokes considers the gravimetric determination of the geoid. Its solution, that is *Stokes' formula* or *Stokes' integral*, see [94] Stokes 1849, is considered one of the most important formulas of physical geodesy and is given as follows

$$N = \frac{R}{4\pi\bar{\Gamma}} \iint\limits_{S} \Delta\Gamma \mathrm{S}(\psi) \mathrm{d}S. \tag{2.71}$$

In (2.71), $\mathrm{S}(\psi)$ denotes the famous *Stokes function*, which can be expressed in a closed form as given, e.g., in [28] Heiskanen&Moritz 1967

$$\mathrm{S}(\psi) = \frac{1}{\sin\frac{\psi}{2}} - 6\sin\frac{\psi}{2} + 1 - 5\cos\psi - 3\cos\psi\ln\left(\sin\frac{\psi}{2} + \sin^2\frac{\psi}{2}\right). \tag{2.72}$$

The argument of Stokes' function, i.e. the angle ψ, is the *spherical distance* between two points on the sphere S. See, e.g., [28] Heiskanen&Moritz 1967 or [57] Kuhn 2000 for the numerical evaluation of the integral formula (2.71) and for the properties of the function (2.72).
In view of the practical solution, one is confronted with the necessity to have the gravity anomaly data available on the sphere S, cf. (2.69), or rather on the geoid. Note, that the geoid in fact corresponds to the sphere S in the spherically approximated case considered above. However, observations are actually not made directly on the geoid but on the physical surface of the Earth. Hence, the measurements have to be reduced in order to obtain boundary values for the geoid. For that purpose, external masses outside the geoid have to be taken into account before the gravity data can be downward continued from the Earth's surface to the geoid. For a complete survey of all possible reductions, that is e.g. the Bouger reduction, the terrain correction, the Poincaré-Prey reduction, the free-air reduction, the isostatic compensation, the Rudzki reduction, the condensation reduction, etc., the interested reader is again referred to, e.g., [28] Heiskanen&Moritz 1967. As is known, most of these reductions involve certain assumptions, e.g., on the density of the masses outside the geoid or on value of the free-air gravity gradient. Not so much from the practical influence but rather from a theoretical point of view this is fairly unsatisfying.

On this account, M.S. Molodensky introduced in 1945 a completely new idea by abandoning the concept of the geoid, see [63],[64] Molodensky 1945,1962. His work has ever since been a starting point for further advances in the theory of the GBVP, see e.g. [68] Moritz 2000 for a valuable compilation of literature references. To name but a few, his research attracted the attention of [55] Krarup 1969, [31] Hörmander 1975, [80] Sansò 1978 and, vicariously, [6] Borre 2006.
M.S. Molodensky's theory is essentially free of any assumptions. However, the mathematical formulation becomes more abstract and more difficult. In short, the problem of Molodensky can be summarized as follows. The aim is the direct gravimetric determination of the physical Earth's surface instead of the geoid. By this means, the corresponding boundary data does not anymore refer to the geoid but to the actual physical surface of the Earth. Consequently, there is no mass density information required. In the following, the focus of this thesis is on Molodensky's problem.

Remark 3 For the sake of completeness it must be pointed out that besides the above mentioned strategy to enforce an exterior BVP in the context of Stokes' problem by means of a sophisticated reductions and compensations measures, an alternative solution strategy is possible. In that case, the problem to determine the geoid is regarded as an interior BVP since the geoid, especially in the continental regions, lies partially inside the masses. Consequently, the effect of the masses outside the geoid must not be removed by computation. In fact, the actual surface gravity data together with the effect of the external masses enter into the solution process. Thus, Poisson's rather than Laplace's equation governs the solution of Stokes problem. Among others, this point of view is shared in the following contributions [16] GRAFAREND 1989; [11],[12] ENGELS 1991,1993 and [34],[35],[36] HOLOTA 1994,1997,2003. Often, for instance within [36] HOLOTA 2003, advantage is taken of the fact that the general solution of Poissons's equation can be split up. That is, it can be expressed as the sum of the general solution of Laplace's equation and of the particular solution of Poisson's equation.

2-5 Contact transformations – introductory remarks

This section is meant to provide an introduction to the subject of contact transformations. A condensed presentation of the theory on contact transformations is of vital importance for the treatment of the GBVP in the framework of the three different gravity space approaches, which will be presented in this work. The underlying theory was already founded in 1868 by the Norwegian mathematician S. Lie. He addressed contact transformations in his famous work about transformation groups, which form the basis of what is today referred to as *Lie algebra*. Lie algebra is often employed for the investigation of geometrical structures and for the solution of differential equation systems. In connection with the latter, it should be pointed out that Lie's numerical integration method based on the well known Lie series is of practical importance in the field of celestial mechanics for the computation of planetary orbits, see e.g. [96] STUMPFF 1974; [87] SCHNEIDER 1979. Due to the outstanding impact of Lie's contributions, the material as presented in the following on contact transformations, is arranged according to his work, e.g. given in [60] LIE 1970. Additionally, the chapter on contact transformations in [9] CARATHEODORY 1956 can be recommended as worthwhile secondary literature.
It is customary in mathematics to use suitable transformations to simplify the problem at hand. Of particular interest is the family of the so-called *contact transformations*, which form a special transformation class for differential equations. Especially, a close relationship to the theory of first order partial differential equations can be constituted, which will also be the focus here. The basic idea of utilizing contact transformations is that the solution set of the primary differential equations changes into the solution set of the transformed differential equations. Thereby, the specific characteristics of contact transformations are such that the derivatives u' of the sought-after functions $u = u(x)$ usually form independent variables after transformation. This peculiarity is motivated by the fact that the original differential equation, formally given by

$$f(x, u, u') = 0\,,$$

is complicated in terms of the derivative u', whereas the transformed differential equation, formally written as

$$F(X, U, U') = 0\,,$$

is simple with regard to the derivative U' of the new unknown function $U = U(X)$. The reason is that the derivatives u' are, in a manner of speaking, absorbed in the new independent variable X. Now, in the context of the next paragraph, which initially focuses on a planar scenario, the background for the theory of contact transformations shall be provided. Based on the plausible findings in the two-dimensional case, the subsequent paragraph gives the general definition of contact transformations as will be required for the formulation of regular gravity space transformations. Finally, a special contact transformation, i.e. the *Legendre transformation*, is discussed. As will be seen later on, the Legendre transformation forms the foundation of F. Sansò's gravity space transformation.

2-5.1 Contact transformations in the plane

A transformation in two dimensions in terms of the two independent variables (x, y) is formally given by

$$x_1 = X(x, y) \quad , \qquad y_1 = Y(x, y)\,. \tag{2.73}$$

The planar transformation (2.73) can be understood as an operation that transfers a point with coordinates (x, y) into a new point with coordinates (x_1, y_1). The points (x, y) and (x_1, y_1) are located in the same plane and refer to the same orthogonal coordinate system. Supposing that not a single point transformation is of interest, but the transformation of a set of points constituting an arbitrary curve C, then each point (x, y) of C, which is formally given by

$$y - \varphi(x) = 0\,, \tag{2.74}$$

can be associated with a particular value y' for the differential quotient

$$\frac{dy}{dx} = y'\,. \tag{2.75}$$

It is commonly known that the derivative y' represents the slope of the corresponding tangent line of C at (x, y) with respect to the x-axis and at the same time determines the common direction of the tangent line and of C in (x, y). Moreover, according to [60] Lie 1970, it follows that the point transformation (2.73) involves not only a transformation of (x, y) into (x_1, y_1), but also at the same time a transformation of the derivative y' into the derivative

$$y_1' = \frac{dy_1}{dx_1} = \frac{dY}{dX}\,. \tag{2.76}$$

Furthermore, S. Lie shows that the transformed derivative y_1' is in fact uniquely determined as a function of (x, y, y')

$$y_1' = y_1'(x, y, y')\,, \tag{2.77}$$

when $X(x, y)$, $Y(x, y)$ and y' are known. Consequently, (2.73) does not only establish the relationship of (x, y) and (x_1, y_1), but also the relationship between y' and y_1'. Strictly speaking, there exists a one-to-one correspondence between the quantities (x, y, y') and the quantities (x_1, y_1, y_1'), which motivated S. Lie to propose the following new transformation with respect to the three independent variables (x, y, y'):

Definition 1 *A contact transformation in the x, y-plane is given by*

$$x_1 = X(x, y, y') \quad , \qquad y_1 = Y(x, y, y') \quad , \qquad y_1' = P(x, y, y')\,, \tag{2.78}$$

so that an identity in the following form

$$dY - PdX = \rho(dy - y'dx) \tag{2.79}$$

is satisfied and that, in addition, the quantity $\rho = \rho(x, y, y')$ is determined to be non-zero.

Now, by reconsidering (2.73) for the time being, geometrical considerations, particularly with regard to the transformation of the derivative y', shall provide a better understanding of the idea of contact transformations. On this account, the tangent line to the curve $y - \varphi(x) = 0$ at a fixed point is considered. With known values for (x, y, y') at this point, the corresponding equation, which all points (η, χ) constituting the tangent line have to satisfy, reads

$$\eta - y = y'(\chi - x)\,. \tag{2.80}$$

Similarly, the tangent line of the transformed curve $y_1 - \varphi_1(x_1) = 0$ at the related point $(x_1 = X, y_1 = Y)$ is given by

$$\eta - y_1 = y_1'(\chi - x_1)\,, \tag{2.81}$$

where (x_1, y_1) is related to (x, y) by means of (2.73) and, as stated above, $y_1' = y_1'(x, y, y')$ also results as a consequence of (2.73). Now, it directly follows from (2.80) and (2.81) that all curves, which have with the curve $y - \varphi(x) = 0$ a point (x, y) and the corresponding tangent line (2.80) in common, transform by means of (2.73) into such curves that have with the curve $y_1 - \varphi_1(x_1) = 0$ the point $(x_1 = X, y_1 = Y)$ and the corresponding tangent line (2.81) in common. In short, by means of the transformation (2.73), all curves that are mutually tangent at a common point are transformed into such curves that are also mutually tangent at a common point, Fig. 2.1. Moreover, besides transforming curves with common tangent lines into curves with common tangent lines, (2.73) transforms all tangent lines related to a common intersection point of different curves into tangent lines related again to a common point, which represents the transformed intersection point of the transformed curves, Fig. 2.2.

Due to the fact that the point transformation (2.73) apparently involves not only the transformation of points (x, y) to points (x_1, y_1), but also the transformation of the corresponding tangent lines, it makes sense to treat the geometrical figure of a point (x, y), and the corresponding tangent line going through it, as compound element. Henceforth, this configuration is referred to as *line element* and its associated coordinates are (x, y, y'). Consequently, it can be stated that each planar point transformation of type (2.73) is connected with a transformation of

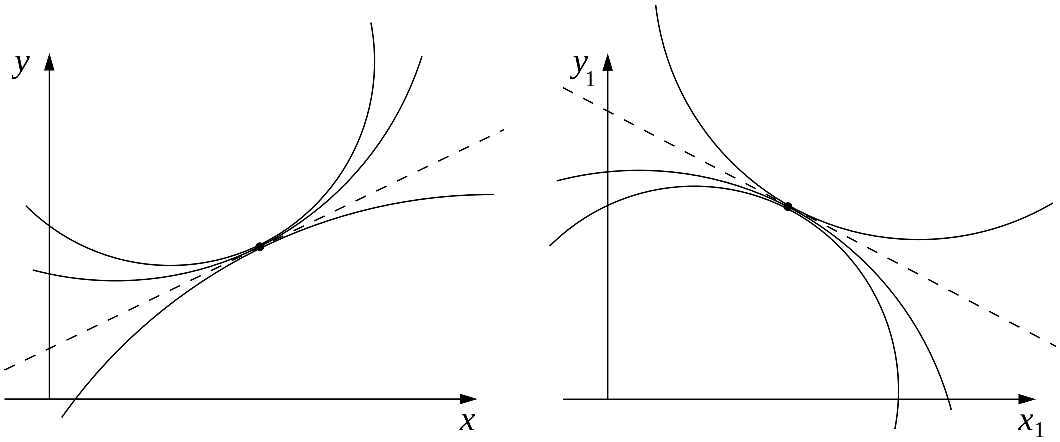


Figure 2.1: Transformation of curves with common tangent lines.

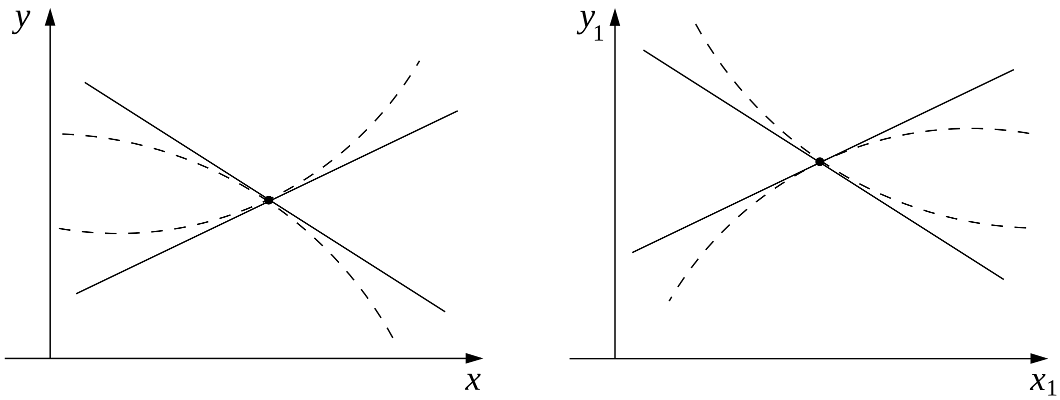


Figure 2.2: Transformation of tangent lines at intersection points of different curves.

such line elements (x, y, y') in the same plane, and that the analytical expression for the transformed line element results due to (2.73) only, as indicated by (2.77). Therefore, (2.73) can be characterized as a point transformation, and also at the same time as a *line element transformation*, when the direction of the tangent line is additionally transformed.

In [60] Lie 1970 it is shown that the line element transformation related to (2.73) is in agreement with (2.78) and satisfies the constituting condition (2.79). Hence, a line element transformation is always a contact transformation also , and therefore from a certain point of view, a descriptive interpretation for (2.78) has been found. That is, in short, contact transformations transform line elements, i.e. positions and directions, rather than points.
Nevertheless, it must be admitted that the class of contact transformations generally comprises a much broader spectrum of transformations. This is since an unlimited number of transformations can be established, which are in accordance with (2.78) and satisfy the condition (2.79). However, only such contact transformations that are also line element transformations, and exhibit the transformation properties as illustrated in Fig. 2.1 and Fig. 2.2, are of importance in the course of this work.

2-5.2 Contact transformations in n - dimensional space

In the general case, that is by abandoning the planar approach, a contact transformation, according to Lie 1970, is given as follows:

Definition 2 *Let be* $\mathbf{x} = [\ x_1 \quad x_2 \quad \dots \quad x_n\ ]$ *and* $\mathbf{p} = [\ p_1 \quad p_2 \quad \dots \quad p_n\ ]$. *Moreover,* $\mathbf{X} = [\ X_1 \quad X_2 \quad \dots \quad X_n\ ]$ *and* $\mathbf{P} = [\ P_1 \quad P_2 \quad \dots \quad P_n\ ]$ *holds, then a contact transformation*

$$\mathbf{X} = \mathbf{X}(\mathbf{x}, u, \mathbf{p}) \quad , \qquad U = U(\mathbf{x}, u, \mathbf{p}) \quad , \qquad \mathbf{P} = \mathbf{P}(\mathbf{x}, u, \mathbf{p}) \tag{2.82}$$

constitutes a diffeomorphism, which maps an open domain $G \subset \mathbb{R}^{2n+1}$ to an open domain $\Omega \subset \mathbb{R}^{2n+1}$, so that an identity of the following form

$$dU - \sum_{j=1}^{n} P_j dX_j = \rho\Big(du - \sum_{j=1}^{n} p_j dx_j\Big) \tag{2.83}$$

is satisfied in G, and that in addition the smooth quantity $\rho = \rho(\mathbf{x}, u, \mathbf{p})$ is determined to be non-zero in G.

In Definition 2, $\mathbf{x}$ denotes the primary coordinate vector and $\mathbf{X}$ the transformed coordinated vector. Similarly, u refers to a scalar function before transformation and U to the same function after transformation. Consequently, the same holds for the quantities $\mathbf{p}$ and $\mathbf{P}$. The meaning of $\mathbf{p}$ and $\mathbf{P}$ is addressed below in more detail.

As already indicated in the beginning of this section, the benefit of such a transformation is that solutions of differential equations are invariant with respect to contact transformations. That is, if $u = u(\mathbf{x})$ represents a solution of the differential equation

$$f(\mathbf{x}, u, \nabla u) = 0\,, \tag{2.84}$$

then $U = U(\mathbf{X})$ represents a solution of the differential equation

$$F(\mathbf{X}, U, \nabla U) = 0\,, \tag{2.85}$$

which results from (2.84) by means of (2.82) and in compliance with the condition (2.83), thereby putting

$$p_j = \frac{\partial u}{\partial x_j}\,, \qquad j = 1, \ldots, n\,. \tag{2.86}$$

Hence, $\mathbf{p} = \nabla u$ holds. Analogously,

$$P_j = \frac{\partial U}{\partial X_j}\,, \qquad j = 1, \ldots, n\,. \tag{2.87}$$

is obtained. Concerning (2.86) and (2.87), it should be noted that according to the nomenclature conventionally used in classical mechanics, p_j and P_j, respectively, are denoted as *generalized momentum variables*. As expected, $\mathbf{p}$ in (2.86) is related to the function u before the transformation, whereas $\mathbf{P}$ in (2.87) is associated with U after the transformation, according to (2.82) and (2.83).

2-5.3 Legendre transformation

This paragraph considers a particular contact transformation, which will be of substantial interest in the course of this work, namely the so-called *Legendre transformation*:

Definition 3 *A Legendre transformation is given by*

$$X_j = p_j \quad , \qquad U = \sum_{j=1}^{n} p_j x_j - u \quad , \qquad P_j = x_j\ , \qquad j = 1, \ldots, n\,. \tag{2.88}$$

Hence, by means of (2.88) the primary or old momentum variables p_j become the transformed or new coordinates X_j and the old coordinates x_j transform into the new momenta P_j. Moreover, the assumption that the Legendre transformation (2.88) indeed constitutes a contact transformation is confirmed by the following reasoning, based on differentiation of (2.88):

$$\begin{aligned} dU &= d\Big(\sum_{j=1}^{n} p_j x_j - u\Big) \\ &= \sum_{j=1}^{n} d(p_j x_j) - du \\ &= \sum_{j=1}^{n} p_j dx_j + \sum_{j=1}^{n} x_j dp_j - du \end{aligned} \tag{2.89}$$

Now, according to (2.88),

$$\sum_{j=1}^{n} x_j dp_j = \sum_{j=1}^{n} P_j dX_j \tag{2.90}$$

holds true. Consequently, by using this relation in (2.89)

$$dU - \sum_{j=1}^{n} P_j dX_j = -\left(du - \sum_{j=1}^{n} p_j dx_j\right) \tag{2.91}$$

is derived. Finally, by comparing (2.91) to (2.83), thereby assuming $\rho = -1$, it can be concluded that (2.88), as a matter of fact, does represent a contact transformation.

Furthermore, due to the striking symmetry of (2.88), the backward transformation can be simply defined as follows:

Definition 4 *The inverse transformation of (2.88) reads as*

$$x_j = P_j \quad , \qquad u = \sum_{j=1}^{n} P_j X_j - U \quad , \qquad p_j = X_j \ , \qquad j = 1, \ldots, n \, . \tag{2.92}$$

Next, a geometrical interpretation of the Legendre transformation is possible by getting back to the planar approach once more. The basic idea is that a curve $C : u = u(x)$, Fig. 2.3 a, is not interpreted as the sum of its constituting points (x, u), but as the envelope of the corresponding tangent lines as illustrated in Fig. 2.3 b. These tangent lines are thereby uniquely defined in terms of associated tangent coordinates (ξ, τ). See Fig. 2.3 a, where the slope and the u-intercept are formally denoted by τ and $-\xi$, respectively. Then, from the equation for the family of these tangent lines, follows the equation of C in terms of tangent coordinates, i.e. $C : \xi = \xi(\tau)$, Fig. 2.3 c.

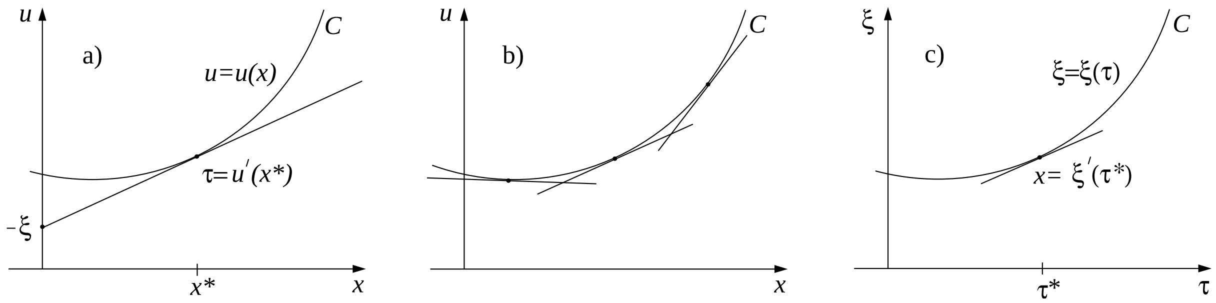


Figure 2.3: Legendre transformation in two dimensions.

An analytical formulation is obtained by starting from the equation

$$u = u(x) \, , \tag{2.93}$$

given for the curve C in position coordinates (x, u). Next, the relation for a corresponding tangent line at a fixed point $(x^\star, u(x^\star))$ is

$$u = \tau x - \xi \, , \tag{2.94}$$

subject to

$$\begin{aligned} \tau &= u'(x^\star) \\ \xi &= x^\star u'(x^\star) - u(x^\star) \, , \end{aligned} \tag{2.95}$$

where the first relation is directly conceivable and where the second relation results from the point-slope equation for straight lines, cf. Fig. 2.3 a.
The family of tangent lines that belong to C, is obtained in terms of the free parameter $x^\star$. The corresponding envelope results from the elimination of $x^\star$ in (2.95) by solving the first expression for $x^\star = x^\star(\tau)$ and substituting it into the second expression to obtain $C : \xi = \xi(\tau)$.

On the other hand, considering the problem in a reverse manner, starting from the known representations $C : \xi = \xi(\tau)$, the family of tangent lines is defined as

$$u = \tau^\star x - \xi(\tau^\star) \, . \tag{2.96}$$

In (2.96), $\tau^\star$ now denotes the free parameter, and consequently by its elimination from (2.96), thereby considering the side condition $\xi'(\tau) = x$, $C : u = u(x)$ is derived.

It should be pointed out that the transition from position coordinates (x, u) to tangent coordinates (τ, ξ), (2.94), is formally identical to the Legendre transformation, as specified in (2.88). This becomes obvious by rearranging (2.94) as follows

$$\xi = \tau x - u = u'x - u \,, \tag{2.97}$$

thereby substituting $\tau = u'$, according to (2.95). Subsequently, applying the relation $p = u'$, (2.86), and using U instead of ξ in (2.97), directly yields (2.88) for $n = 1$. By comparing Fig. 2.3 a) and c), it is observed that the Legendre transformation indeed transforms line elements (x, u, u') into line elements (τ, ξ, ξ'). Thus, the Legendre transformation belongs to the family of contact transformations.

In conclusion it can be stated that the benefit of contact transformations, specifically Legendre transformations, is the property to transform first order partial differential equations, see (2.84), (2.85). It is worth also mentioning that the Legendre transformation is commonly used in other fields such as classical mechanics, celestial mechanics and thermodynamics. A prominent example in mechanics is that the Hamiltonian represents the Legendre transform of the Lagrangian, replacing generalized velocity variables with generalized momentum variables, see e.g. [72] Päsler 1968.

Chapter 3

Vectorial free GBVP in geometry space

The task of physical geodesy can be summarized as follows: determination of the external gravity field of the Earth and the geoid. Before the availability of modern observational methods such as airborne or spaceborne techniques this task has mainly been pursued by boundary value approaches enabling the determination of the gravity field, on and outside the Earth's surface, from ground gravity data. Such BVPs aiming at the determination of the physical shape of the Earth and the terrestrial gravity field are referred to as geodetic BVPs. They are different from the classical BVPs in potential theory, cf. Section 2-3, in two ways. Firstly, the boundary surface of the BVPs in potential theory is assumed to be known, which is in contrast to the BVPs in physical geodesy where the boundary surface is usually to be determined. Secondly, the boundary data type is different. Whereas the BVPs discussed in Section 2-3 refer to functions of the *gravitational* potential, the BVP introduced in this chapter is based on functions of the *gravity* potential.

3-1 Classification of the boundary value problems of physical geodesy

Nowadays, the term *geodetic boundary value problem* (GBVP), which was originally restricted to Molodensky's and Stokes' problems, comprises a still increasing class of boundary value problems connected with the Laplace-Poisson partial differential equation. Depending on the type of boundary data, and on the type and number of unknown functions to be solved from geodetic observational data, it is important to distinguish between several formulations of the GBVP. Based on terrestrial geodetic observables such as potential data or potential differences, modulus of the gravity vector and astronomical latitude and longitude, three classical versions of the GBVP can be formulated: first of all, the *fixed* or *gravimetric* GBVP, involving the assumption of a completely known boundary surface, secondly, the *scalar free* and, thirdly, the *vectorial free* GBVP, where either only the vertical component or the full three dimensional boundary position vector has to be determined from the given boundary data simultaneously with the gravity potential, e.g. [22] Heck 1991.
In recent years new observational techniques have created non-classical types of boundary data. As a consequence, new BVP versions such as the *gradiometric*, e.g. [17] Heck 1979; [14] van Gelderen 2001; [62] Martinec 2003, *mixed* or *altimetry-gravimetry*, e.g. [32],[33] Holota 1983; [49] Keller 1996 and *overdetermined* BVP, e.g. [75] Sacerdote 1985; [74] Rummel 1989; [29] Hirsch 1996, have to be considered. However, these new BVP types will not be discussed in this work.
In general, all GBVPs are *nonlinear* problems since the observables depend on the unknown functions in a nonlinear way. Therefore, investigation of existence and uniqueness properties of the corresponding GBVP solution, in addition to its numerical evaluation, implies considerable mathematical and computational difficulties, e.g. [31] Hörmander 1975; [66] Moritz 1980; [85] Sansò 1981; [105],[106] Witsch 1985,1986; [48] Keller 1987.
For practical purposes, nonlinear GBVPs are linearized with respect to a reference or normal potential, see e.g. [31] Hörmander 1975. Additionally, the vectorial free and the scalar free GBVPs require the choice of a mathematical reference surface approximating the physical surface of the Earth. Depending on the choice of reference surface, the associated BVP is referred to as Molodensky's or Stokes' problem. Naturally, the linearized GBVPs represent *oblique* derivative problems, since the potential gradient is known along the plumb lines, which are usually not orthogonal to the physical Earth's surface or the reference surface. Thus, the process of BVP linearization is generally followed by further approximations in order to modify the corresponding boundary condition. To begin with, by means of the so-called *spherical approximation*, the boundary operator is simplified. This step basically

involves the use of spherical coordinates and adoption of an isotropic normal potential, cf. (2.40), in order to revise the form of the boundary operator. The last two steps in the hierarchy of approximations address the problem of reducing the complexity of the underlying boundary surface. The *planar approximation* and the *constant radius approximation* formally transfer the boundary data onto a sphere. Hence, it can be noted that the oblique derivative problem has been transformed into a simple normal derivative problem. A comprehensive overview of the classification of the above approximations can be found, e.g., in [26] HECK 2003.
Despite the fact that due to deficiencies concerning availability and accuracy of astronomic observations, the practical significance of the vectorial free GBVP is questioned, e.g. in [19],[25] HECK 1988,1997, a closer investigation of this BVP type shall first be conducted in Section 3-3. The reason is that the vectorial free GBVP represents the most general form of the classical geodetic BVP types. Basic findings made for this BVP type can be carried over to the fixed and scalar free problem. Therefore the aspect of linearization is treated in the context of the vectorial free GBVP in Section 3-4. Accordingly, spherical and constant radius approximation are elaborated for the vectorial free GBVP in Section 3-6 and Section 3-7.
For the sake of completeness it should be pointed out that detailed descriptions of the scalar free GBVP and the fixed GBVP can be found in the following contributions. Theoretical aspects and investigations on existence and uniqueness of the scalar free GBVP, which is also referred to as the *geodetic variant of Molodensky's problem*, have been evaluated, e.g., in [76] SACERDOTE 1986; [70],[71] OTERO 1987,1999 and [21] HECK 1989. The proof concerning existence and uniqueness of the fixed gravimetric BVP can be found, for example in [54] KOCH 1972 and [5] BJERHAMMAR 1983. Solutions to the problem are given in [53] KOCH 1971; [92] STOCK 1983 or [20] HECK 1989. Numerical solutions via the boundary element method can be found e.g. in [51],[52] KLEES 1992,1997 and [58] LEHMANN 1997.

Prior to the discussion of the vectorial free GBVP in more detail, some simplifying assumptions for the Earth system have to be made.

3-2 Idealized Earth model assumptions

The respective physical-mathematical Earth model when dealing with GBVPs can be formulated as follows, e.g. [66] MORITZ 1980. The Earth is assumed to behave like a rigid, non-deformable body, uniformly rotating with known angular velocity ω about a space- and body-fixed axis, passing through the Earth's center of mass (CoM). This CoM represents the origin 0 of a Cartesian coordinate system, the x_3-axis coinciding with the axis of rotation. This work deals with a gravity field produced solely by the Earth's inner masses, i.e. all attracting mass elements are located in the interior of the closed boundary surface σ, which represents the Earth's surface. The gravity potential and the gravity vector must be corrected for the direct gravitational effect of other bodies, i.e. mainly moon and sun, and for the attraction of the atmosphere. Furthermore, indirect tidal effects of celestial bodies as well as geodynamic phenomena are considered to be accounted for since we want to deal with a rigid Earth. In short, a steady field of a steady state Earth is considered henceforth.

3-3 The vectorial free GBVP

As mentioned above, the treatment of a particular BVP, namely the vectorial free GBVP, which, from now on, is always meant if simply addressed by GBVP, is the main focus of this work. First, it shall be formally specified:

Definition 5 *The geodetic boundary value problem is identified as follows: let σ be a closed orientable and sufficiently smooth boundary surface of a simply connected region with unknown embedding and metric. Astronomical latitude Φ and longitude Λ serve as the corresponding surface parameters on σ. The data, i.e. gravity potential values and gravity vectors, are given on σ*

$$\begin{aligned} w : \sigma &\rightarrow \mathbb{R} \\ \tilde{\mathbf{\Gamma}} : \sigma &\rightarrow \mathbb{R}^3 \end{aligned}$$

and to be found are: (i) a real function $W(\mathbf{x}) : ext\ \sigma \rightarrow \mathbb{R}$, such that

$$\Delta W(\mathbf{x}) = 2\omega^2\ , \quad \mathbf{x} \in ext\ \sigma \tag{3.1}$$

$$W\big|_{\sigma} = w \tag{3.2}$$

$$\nabla W\big|_{\sigma} = \tilde{\mathbf{\Gamma}} \tag{3.3}$$

holds, and (ii) the surface σ that represents the physical Earth surface.

The nonlinear BVP introduced by Definition 5 is sometimes referred to as the *astronomical variant* of the free GBVP, see [22] HECK 1991. The total number of unknown quantities related to this BVP is *four*: the (external) gravity potential $W(\mathbf{x}) : ext\ \sigma \to \mathbb{R}$ of the Earth, and the three components of a position vector $\mathbf{x}\big|_{\sigma}$ describing the unknown geometry of the Earth's boundary σ. This setup of four unknowns has to be counterbalanced by the observation of (at least) four different observables, namely the modulus of the gravity vector $\tilde{\Gamma} = \|\tilde{\mathbf{\Gamma}}\|$ from gravimetric observations, astronomical latitude Φ and longitude Λ from astro-geodetic measurements and ground gravity potential values w from leveling.

Remark 4 It must be pointed out that the potential values w are practically deduced in terms of differences between the potential at the geoid and the potential at the respective points. These potential differences, also referred to as *geopotential numbers*, are derived from spirit leveling combined with gravity measurements. Hence, potential values w can only be determined up to an unknown constant, i.e. the potential value at the geoid. According to [28] HEISKANEN&MORITZ 1967, the geoidal potential can be scaled with respect to the adopted normal potential by measuring at least one distance and also the astronomical coordinates of its end points. In fact, it is state-of-the-art that nowadays an unknown constant is determined for each continent, see e.g. [10] COLOMBO 1980 or [27] HECK 2004. For the sake of simplicity, this peculiarity in determining absolute potential values from relative potential values is disregarded. Henceforth, the assumption is that absolute potential values are directly observable.

3-4 Linearization of the vectorial free GBVP

Linearization, as already indicated, is generally the first step in the process of solving BVPs and involves introducing suitable known approximate quantities and applying Taylor's theorem. Pending further notice, the way of presenting the subject of linearizing the vectorial free GBVP follows the works of [64] MOLODENSKY 1962 and [65] MORITZ 1977. On this account it should be pointed out that the term *Molodensky's problem* is often used synonymously for GBVP in literature. The material devoted to the aspect of linearizing the GBVP or rather Molodensky's problem is organized as follows. To begin with, a first Lemma establishes a linearized relation between potential disturbances and potential anomalies. A second Lemma addresses the problem of finding a linearized relationship of the gravity disturbance vector and the gravity anomaly vector. A theorem based on these findings presents the fundamental boundary condition of the linearized Molodensky problem, which is explicitly stated in a final definition.
In order to get started, the true gravity potential W is approximated by the normal gravity potential W_0. In general, a Somigliana-Pizzetti type of normal gravity potential is chosen for W_0. Furthermore, the real Earth surface is approximated by a known surface, the so-called *telluroid.* In order to serve as a suitable reference for the Earth's surface σ, the telluroid surface Σ has to be introduced in such a way that points P of σ and points Q of Σ are uniquely related, see Fig. 3.1. This relationship, more precisely the vector $\boldsymbol{\zeta}$ pointing from Q to P, referred to as *vectorial height anomaly* or *position correction vector*, is unknown and must be determined in the course of the BVP solution.

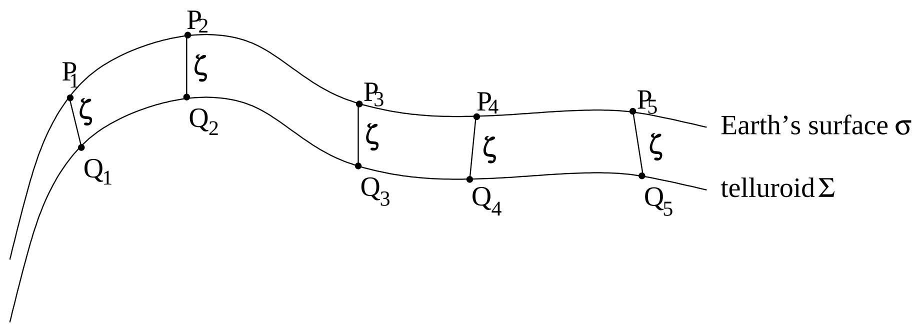


Figure 3.1: A mathematical reference surface approximating the physical surface of the Earth.

For reasons that will become clear in a later section, the telluroid surface Σ is defined in a rigorous way as follows:

Definition 6 *Assume the true gravity potential and the direction of the true gravity vector to be given, i.e. W, Φ and Λ are known at P, then the points Q forming the telluroid Σ must satisfy the following three conditions*

$$W_0(Q) = W_0\big|_\Sigma = W(P) = W\big|_\sigma \quad ; \quad \hat{\phi}_Q = \Phi_P \quad ; \quad \hat{\lambda}_Q = \Lambda_P\,. \tag{3.4}$$

That is, the normal potential W_0, the normal latitude $\hat{\phi}$ and the normal longitude $\hat{\lambda}$ at Q equal the true potential W, the astronomical latitude Φ and the astronomical longitude Λ at P, respectively. Note that $\hat{\phi}$ and $\hat{\lambda}$ determine the direction of the normal gravity vector $\mathbf{\Gamma}_0$, cf. (2.36), whereas Φ and Λ fix the direction of the actual gravity vector $\mathbf{\Gamma}$, see (2.25). In other words, the normal vertical on the telluroid and the local astronomical zenith on the Earth coincide. Hence, (3.4) represents an isoparametric, more specifically isozenithal, telluroid mapping, see e.g. [15] Grafarend 1978. Since the triplet latitude, longitude and potential is usually referred to as *Marussi coordinates*, the resulting telluroid is also called the *Marussi telluroid*. A comparison of several other telluroid realizations, such as the *gravimetric telluroid* mapping, which will be applied later on (Definition 18), is given, e.g., in [18] Heck 1986.

Thus the necessary preparations are met to linearize the BVP given according to Definition 5 starting from the following lemma:

Lemma 2 *Given that $\boldsymbol{\zeta}$, δW and $\nabla\delta W$ are small enough quantities, and given that in accordance with the concept of linear approximation the terms of second and higher order are neglected, then the relationship*

$$\delta W\big|_\Sigma + \mathbf{\Gamma}_0^\top\big|_\Sigma \boldsymbol{\zeta} = \Delta w\,, \tag{3.5}$$

between gravity potential disturbances and gravity potential anomalies holds.

Proof. Starting from the definition given for the gravity potential anomaly in (2.46)

$$W\big|_\sigma - W_0\big|_\Sigma = \Delta w \tag{3.6}$$

and rearranging the expression for the gravity potential disturbance (2.47) into

$$W\big|_\sigma = \delta W\big|_\sigma + W_0\big|_\sigma \tag{3.7}$$

leads immediately to

$$\delta W\big|_\sigma + W_0\big|_\sigma - W_0\big|_\Sigma = \Delta w \tag{3.8}$$

by substituting (3.7) in (3.6). Denoting the separation of a point P at the topography σ from a directly related point Q on the telluroid Σ as *vectorial height anomaly* $\boldsymbol{\zeta}$, and expanding $W_0\big|_\sigma$ at $W_0\big|_\Sigma$ into a Taylor series, thereby omitting higher order terms, results in

$$W_0\big|_\sigma = W_0\big|_\Sigma + \nabla W_0^\top\big|_\Sigma \boldsymbol{\zeta} = W_0\big|_\Sigma + \mathbf{\Gamma}_0^\top\big|_\Sigma \boldsymbol{\zeta}\,. \tag{3.9}$$

Inserting (3.9) in (3.8) gives

$$\delta W\big|_\sigma + \mathbf{\Gamma}_0^\top\big|_\Sigma \boldsymbol{\zeta} = \Delta w\,. \tag{3.10}$$

Similarly to (3.9), the Taylor expansion of $\delta W\big|_\sigma$ at $\delta W\big|_\Sigma$ is given by

$$\delta W\big|_\sigma = \delta W\big|_\Sigma + \nabla\delta W^\top\big|_\Sigma \boldsymbol{\zeta} = \delta W\big|_\Sigma + \delta\mathbf{\Gamma}^\top\big|_\Sigma \boldsymbol{\zeta}\,. \tag{3.11}$$

From (3.11), according to standard textbooks, e.g. [66] Moritz 1980, it follows

$$\delta W\big|_\sigma \cong \delta W\big|_\Sigma\,, \tag{3.12}$$

since both $\nabla\delta W$, or rather $\delta\mathbf{\Gamma}$, and $\boldsymbol{\zeta}$ represent small quantities of first order and therefore their product is of order two and thus negligible in the linear case. Furthermore, consistent with the linear approximation,

$$\nabla\delta W\big|_\sigma = \delta\mathbf{\Gamma}\big|_\sigma \cong \nabla\delta W\big|_\Sigma = \delta\mathbf{\Gamma}\big|_\Sigma \tag{3.13}$$

can be deduced directly from (3.12) with (2.52). Using the equality (3.12) and substituting it in (3.10) finally leads to

$$\delta W\big|_\Sigma + \mathbf{\Gamma}_0^\top\big|_\Sigma \boldsymbol{\zeta} = \Delta w\,. \qquad \diamondsuit$$

Remark 5 It should be pointed out that the assertion of Lemma 2 must be regarded against the background of the question whether the deviation of the disturbing gravity potential at the telluroid from the same quantity at the surface of the Earth can be considered sufficiently small, so that the above linearization is feasible. The answer is connected in turn with the underlying quality of the normal gravity potential, as well as with the adequacy of the defined telluroid surface. As far as these matters are concerned, it can be stated that adequate approximations in both cases are indeed available. E.g. in [18] HECK 1986, it is shown for a Somigliana-Pizzetti type of normal gravity potential and for several telluroid mappings that typical separations between the telluroid surfaces and the Earth's surface are about 100 m.

Lemma 3 *Under the same assumptions as in Lemma 2, that is $\boldsymbol{\zeta}$, δW and $\nabla\delta W$ are small enough quantities and higher order terms are neglected, the following relationship*

$$\delta\mathbf{\Gamma}\big|_\Sigma + \mathbf{M}_0\big|_\Sigma\boldsymbol{\zeta} = \Delta\mathbf{\Gamma} \tag{3.14}$$

between the gravity disturbance and gravity anomaly vector is true.

Proof. In an analogous manner to the modus operandi in the line of argument for Lemma 2, re-ordering (2.48), defining the gravity anomaly vector, results in

$$\mathbf{\Gamma}\big|_\sigma - \mathbf{\Gamma}_0\big|_\Sigma = \Delta\mathbf{\Gamma}\,. \tag{3.15}$$

Next, solving the expression for the gravity disturbance vector (2.49) for $\mathbf{\Gamma}\big|_\sigma$

$$\mathbf{\Gamma}\big|_\sigma = \delta\mathbf{\Gamma}\big|_\sigma + \mathbf{\Gamma}_0\big|_\sigma\,, \tag{3.16}$$

and inserting (3.16) into (3.15) gives

$$\delta\mathbf{\Gamma}\big|_\sigma + \mathbf{\Gamma}_0\big|_\sigma - \mathbf{\Gamma}_0\big|_\Sigma = \Delta\mathbf{\Gamma}\,. \tag{3.17}$$

Following (3.9) using $\mathbf{\Gamma}_0$ instead of W_0, the Taylor expansion of $\mathbf{\Gamma}_0\big|_\sigma$ at $\mathbf{\Gamma}_0\big|_\Sigma$ is deduced

$$\mathbf{\Gamma}_0\big|_\sigma = \mathbf{\Gamma}_0\big|_\Sigma + \nabla\mathbf{\Gamma}_0\big|_\Sigma\boldsymbol{\zeta}\,. \tag{3.18}$$

By means of the matrix $\mathbf{M}_0$ of the second-order partial derivatives of the normal gravity potential W_0, cf. (2.37),

$$\mathbf{M}_0 = \left[M^0_{ij}\right] = \left[\nabla(\nabla W_0)\right] = \left[\frac{\partial W_0}{\partial x_i \partial x_j}\right] = \nabla\mathbf{\Gamma}_0 = \left[\frac{\partial \Gamma^0_i}{\partial x_j}\right]\,,$$

the relationship (3.18) transforms into

$$\mathbf{\Gamma}_0\big|_\sigma = \mathbf{\Gamma}_0\big|_\Sigma + \mathbf{M}_0\big|_\Sigma\boldsymbol{\zeta}\,. \tag{3.19}$$

Using (3.19) together with (3.17),

$$\delta\mathbf{\Gamma}\big|_\sigma + \mathbf{M}_0\big|_\Sigma\boldsymbol{\zeta} = \Delta\mathbf{\Gamma} \tag{3.20}$$

can be determined. Finally, from (3.20) follows with the identity in (3.13)

$$\delta\mathbf{\Gamma}\big|_\Sigma + \mathbf{M}_0\big|_\Sigma\boldsymbol{\zeta} = \Delta\mathbf{\Gamma}\,. \qquad \diamondsuit$$

Lemma 2 and Lemma 3 form the basic relations for the considerations made in the following theorem:

Theorem 1 *By means of Lemma 2 and Lemma 3, and under the same assumptions as before, the fundamental boundary condition of the linearized Molodensky problem is derived as*

$$\left(\delta W + \mathbf{m}^\top\delta\mathbf{\Gamma}\right)\Big|_\Sigma = \Delta w + \mathbf{m}^\top\big|_\Sigma\Delta\mathbf{\Gamma}\,, \tag{3.21}$$

with the vector $\mathbf{m}\big|_\Sigma$ given by

$$\mathbf{m}\big|_\Sigma = -\mathbf{M}_0^{-1}\big|_\Sigma\mathbf{\Gamma}_0\big|_\Sigma\,. \tag{3.22}$$

Proof. At first, (3.14) is solved for $\boldsymbol{\zeta}$ under the assumption that $\mathbf{M}_0$ is invertible

$$\boldsymbol{\zeta} = \mathbf{M}_0^{-1}\big|_\Sigma \left(\Delta\boldsymbol{\Gamma} - \delta\boldsymbol{\Gamma}\big|_\Sigma\right) , \tag{3.23}$$

and is substituted into (3.5)

$$\left(\delta W - \boldsymbol{\Gamma}_0^\top \mathbf{M}_0^{-1}\delta\boldsymbol{\Gamma}\right)\Big|_\Sigma = \Delta w - \boldsymbol{\Gamma}_0^\top\big|_\Sigma \mathbf{M}_0^{-1}\big|_\Sigma \Delta\boldsymbol{\Gamma} .$$

Finally, defining the vector $\mathbf{m}\big|_\Sigma$

$$\mathbf{m}\big|_\Sigma = -\left(\boldsymbol{\Gamma}_0^\top\big|_\Sigma \mathbf{M}_0^{-1}\big|_\Sigma\right)^\top = -\mathbf{M}_0^{-1}\big|_\Sigma \boldsymbol{\Gamma}_0\big|_\Sigma$$

leads with (3.13) to

$$\left(\delta W + \mathbf{m}^\top \delta\boldsymbol{\Gamma}\right)\Big|_\Sigma = \Delta w + \mathbf{m}^\top\big|_\Sigma \Delta\boldsymbol{\Gamma} \qquad \diamondsuit$$

Note that (3.21) is a generalization of the *fundamental equation of physical geodesy* derived in the context of the Stokes' problem. Eq. (3.21), which holds on the telluroid Σ, constitutes the *fundamental boundary condition* of the *linearized* Molodensky problem. It is usually assumed that the gravity potential anomaly Δw and the gravity anomaly vector $\Delta\boldsymbol{\Gamma}$ are given. Thus, by means of the linear combination

$$f := \Delta w + \mathbf{m}^\top\big|_\Sigma \Delta\boldsymbol{\Gamma}$$

of the two observables Δw and $\Delta\boldsymbol{\Gamma}$, and the identity $\nabla\delta W = \delta\boldsymbol{\Gamma}$, (3.21) takes the form

$$\left(\delta W + \mathbf{m}^\top \nabla\delta W\right)\Big|_\Sigma = f , \tag{3.24}$$

so that (3.24) can be identified as an oblique-derivative problem, see Section 2-3.2.

Consequently, the linear variant of the GBVP, also denoted as the linearized Molodensky's problem, is summarized in the following definition:

Definition 7 *The vector-valued linear geodetic boundary value problem is concerned with the following problem: gravity potential anomalies and gravity anomaly vectors are given on the telluroid* Σ

$$\begin{aligned} \Delta w : \Sigma &\rightarrow \mathbb{R} \\ \Delta\boldsymbol{\Gamma} : \Sigma &\rightarrow \mathbb{R}^3 \end{aligned}$$

and needed to be found is a real function $\delta W(\mathbf{x}) : ext\ \Sigma \rightarrow \mathbb{R}$*, such that*

$$\Delta\delta W(\mathbf{x}) = 0 , \quad \mathbf{x} \in ext\ \Sigma \tag{3.25}$$

$$\left(\delta W + \mathbf{m}^\top \nabla\delta W\right)\Big|_\Sigma = f \tag{3.26}$$

$$f = \Delta w + \mathbf{m}^\top\big|_\Sigma \Delta\boldsymbol{\Gamma} \tag{3.27}$$

is satisfied.

After having solved Laplace's equation $\Delta\delta W(\mathbf{x}) = 0$ under the boundary condition (3.26) for the disturbing gravity potential $\delta W(\mathbf{x})$, the vectorial height anomaly $\boldsymbol{\zeta}$ must be determined next. In fact, $\boldsymbol{\zeta}$, which relates the unknown Earth surface σ to its known approximation, that is to the telluroid Σ as illustrated in Fig. 3.1, is derived from (3.23), i.e.

$$\boldsymbol{\zeta} = \mathbf{M}_0^{-1}\big|_\Sigma \left(\Delta\boldsymbol{\Gamma} - \nabla\delta W\big|_\Sigma\right) . \tag{3.28}$$

In general, e.g. [66] Moritz 1980, (3.28) is considered as the generalization of Bruns' formula, cf. (2.70).

3-5 Preliminaries for the spherical approximation of the vectorial free GBVP

In order to solve the BVP given in Definition 7, some simplifications regarding the underlying boundary condition are made. For convenience, the label $\big|_\Sigma$, indicating that the corresponding boundary surface is established by the telluroid surface Σ, is omitted throughout this section. Evaluation of the boundary condition, as presented so far according to (3.26), involves matrix-vector operations to compute the vector $\mathbf{m}$, cf. (3.22). A first lemma, in which the Cartesian components of the normal gravity vector are introduced as new coordinates, reduces the complexity of generating the boundary condition. Instead of matrix-vector products necessary so far, only vector-vector calculations appear henceforth. Furthermore, this modification allows, in the context of the next lemma, to achieve a formulation for the boundary condition, which only makes use of simple scalar-valued operations by introducing so-called *quasi-spherical coordinates*. A third lemma, in which the arc length of an isozenithal line is adopted as a main variable, together with the fourth and last lemma, which modifies the expression of the observational side (3.27), yields the final form of the boundary condition. In its final form, which is still rigorously valid, the basic boundary condition of the linearized Molodensky problem is well suited for geometrical interpretations and the intended further approximations. In addition, it should be pointed out that all modifications applied are based on the contributions of [64] MOLODENSKY 1962 and [65] MORITZ 1977 and leave the boundary condition unaltered in a rigorous sense, i.e. no further approximation is involved.

To begin with, new coordinates are introduced:

Definition 8 *Let the new coordinates p_i be defined by the Cartesian components of the normal gravity vector*

$$\mathbf{p} = [p_i] := \mathbf{\Gamma}_0 = \left[\Gamma_i^0\right] \qquad i = 1,2,3\,. \tag{3.29}$$

Remark 6 At this point, attention should already be drawn to the later Section 4-3, where, in the context of Definition 14, the idea of utilizing the force vector of the Earth's gravity field to define new coordinates is resumed.

Lemma 4 *Reformulation of the boundary condition (3.26) in terms of the new coordinates* $\mathbf{p}$ *yields*

$$\delta W - \mathbf{p}^\top \nabla_{\mathbf{p}} \delta W = \delta W - p_i \frac{\partial \delta W}{\partial p_i} = f\,. \tag{3.30}$$

Proof. Since the coordinates p_i are introduced in such a way that $p_i = \Gamma_i^0$ holds, it follows directly that

$$\mathbf{M}_0 = \left[M_{ij}^0\right] = \left[\frac{\partial p_i}{\partial x_j}\right] \qquad i,j = 1,2,3$$

is true for the matrix $\mathbf{M}_0$ declared in (2.26). Hence, the matrix $\mathbf{M}_0$ represents the Jacobian matrix of the coordinate transformation (3.29). Consequently, the vector $\mathbf{m}$ (keeping $\big|_\Sigma$ in mind), introduced in (3.22), becomes

$$\mathbf{m} = [m_i] = -\mathbf{M}_0^{-1}\mathbf{p} = -\left[\frac{\partial p_i}{\partial x_j}\right]^{-1} [p_j] \qquad i,j = 1,2,3\,, \tag{3.31}$$

using index notation. According to Lemma 1, the inverse of the Jacobian matrix $\mathbf{M}_0$ is simply the Jacobian matrix of the inverse transformation

$$\left[\frac{\partial p_i}{\partial x_j}\right]^{-1} = \left[\frac{\partial x_i}{\partial p_j}\right]\,,$$

(3.31) can be written as

$$m_i = -\frac{\partial x_i}{\partial p_j} p_j \qquad i,j = 1,2,3\,. \tag{3.32}$$

Next, the vector product $\mathbf{m}^\top\big|_\Sigma \nabla \delta W$ in (3.26) is evaluated. Insertion of (3.32) in (3.26) and applying the chain rule of differential calculus, again using index notation and omitting $\big|_\Sigma$, yields

$$\mathbf{m}^\top \nabla \delta W = m_i \frac{\partial \delta W}{\partial x_i} = -\frac{\partial \delta W}{\partial x_i}\frac{\partial x_i}{\partial p_j} p_j = -\frac{\partial \delta W}{\partial p_j} p_j\,. \tag{3.33}$$

Hence, with (3.33), (3.26) becomes

$$\delta W - p_i \frac{\partial \delta W}{\partial p_i} = f\,. \qquad \diamondsuit$$

A comparison of this representation of the boundary condition to the one in (3.26) shows that the explicit setup of the vector $\mathbf{m}$, which involves a matrix-vector product, see (3.22) or rather (3.31), and its application to the gravity disturbance vector $\nabla \delta W$, is substituted by a simple scalar product of two vectors, the new coordinate vector $\mathbf{p}$ and the gradient of the disturbing gravity potential with respect to **p**.

Next, by introducing so-called *quasi-spherical coordinates*, further simplification of the boundary condition (3.30) is achieved:

Definition 9 *Expressing the curvilinear coordinates (p_1, p_2, p_3) given according to Definition 8 in terms of quasi-spherical coordinates (ρ, ϕ, λ) reads as*

$$p_1 = -\frac{1}{\rho^2}\cos\phi\cos\lambda \quad ; \quad p_2 = -\frac{1}{\rho^2}\cos\phi\sin\lambda \quad ; \quad p_3 = -\frac{1}{\rho^2}\sin\phi\,, \tag{3.34}$$

Lemma 5 *In quasi-spherical coordinates the fundamental boundary condition of the linearized Molodensky problem reduces to*

$$2\delta W + \rho\frac{\partial\delta W}{\partial\rho} = 2f\,. \tag{3.35}$$

Proof. The partial derivative $\frac{\partial\delta W}{\partial\rho}$ can be formally expanded into

$$\frac{\partial\delta W}{\partial\rho} = \frac{\partial\delta W}{\partial p_i}\frac{\partial p_i}{\partial\rho} \tag{3.36}$$

by application of the chain rule. A closer look at the partial derivatives $\frac{\partial p_i}{\partial\rho}$ yields

$$\frac{\partial p_i}{\partial\rho} = -\frac{2p_i}{\rho}\,. \tag{3.37}$$

Insertion of (3.37) into (3.36)

$$\frac{\partial\delta W}{\partial\rho} = -\frac{2}{\rho}p_i\frac{\partial\delta W}{\partial p_i}$$

and rearranging the resulting expression leads to

$$-p_i\frac{\partial\delta W}{\partial p_i} = \frac{\rho}{2}\frac{\partial\delta W}{\partial\rho}\,. \tag{3.38}$$

Hence, substitution of (3.38) into (3.30) proves that the assertion of Lemma 5 is true

$$2\delta W + \rho\frac{\partial\delta W}{\partial\rho} = 2f\,. \qquad \diamond$$

Now, the resulting boundary condition (3.35) exhibits the favorable property of being a purely scalar-valued relation. Furthermore, expressing the gravity coordinates p_i in terms quasi-spherical coordinates (ρ, ϕ, λ) allows for the following considerations. Since the vector $\mathbf{p} = [p_i]$ represents nothing else than the normal gravity vector, its modulus is $\|\mathbf{p}\| = \Gamma_0$ and the identity $\Gamma_0 = \frac{1}{\rho^2}$ can be deduced immediately from the definition of the quasi-spherical coordinates (3.34) by comparison. Thus, the factor $\frac{1}{\rho^2}$ simply equals the modulus of the normal gravity vector Γ_0. Since the normal gravity vector is pointing towards the geocenter, the negative sign is mandatory since $\frac{1}{\rho^2}$ is always positive. If the reference ellipsoid is degraded to a reference sphere, the magnitude of the gravity vector Γ_0 is only radially dependent and $\frac{1}{\rho^2}$ becomes proportional to the radius. In a sense, (ρ, ϕ, λ) represent spherical coordinates, hence the name *quasi*-spherical coordinates in the general case. Furthermore, the partial derivative $\frac{\partial\delta W}{\partial\rho}$ in (3.35) means differentiation of the disturbing gravity potential with respect to the coordinate ρ, while the coordinates (ϕ, λ) remain constant. This means, since by definition (3.34), (ϕ, λ) determine the direction of the normal gravity vector $\mathbf{\Gamma}_0$, differentiation takes place along a line where all normal gravity vectors are parallel. Such a line is referred to as *isozenithal*. An important property of isozenithals is that they coincide with the plumb lines when those are straight lines. This situation exists when the underlying mass distribution of a body is considered radially symmetric, e.g. in case of a homogeneous non-rotating sphere. However, as the normal plumb line curvature is quite small, isozenithals and plumb lines are not very different, cf. [97] Sünkel 1978.

The use of isozenithals provides the opportunity to simplify (3.35) yet another time by:

Lemma 6 *Let τ denote the arc length of an isozenithal line with respect to the normal gravity field, then the boundary condition (3.35) takes the form*

$$\frac{\partial \delta W}{\partial \tau} - \frac{1}{\Gamma_0}\frac{\partial \Gamma_0}{\partial \tau}\delta W = -\frac{1}{\Gamma_0}\frac{\partial \Gamma_0}{\partial \tau} f \, . \tag{3.39}$$

Proof. The new variable τ stands for the arc length of an isozenithal line. Therefore, $\frac{\partial}{\partial \tau}$ is a derivative along the isozenithal of the normal gravity field in the same way as $\frac{\partial}{\partial \rho}$. Hence, the two partial derivatives can only differ by a scalar factor C

$$\frac{\partial}{\partial \tau} = C\frac{\partial}{\partial \rho} \, . \tag{3.40}$$

The proportionality factor C is obtained by applying (3.40) to the modulus of the normal gravity vector Γ_0

$$\frac{\partial \Gamma_0}{\partial \tau} = C\frac{\partial \Gamma_0}{\partial \rho} \, . \tag{3.41}$$

As already discussed above, the following relationship

$$\|\mathbf{p}\| = \Gamma_0 = \frac{1}{\rho^2}$$

follows directly from (3.29) and (3.34). Thus, the derivative $\frac{\partial \Gamma_0}{\partial \rho}$ can be evaluated

$$\frac{\partial \Gamma_0}{\partial \rho} = \frac{\partial}{\partial \rho}\left(\frac{1}{\rho^2}\right) = -\frac{2}{\rho^3} = -\frac{2\Gamma_0}{\rho} \, . \tag{3.42}$$

Solving (3.41) for the proportionality factor C and utilizing (3.42) yields

$$C = \frac{\partial \Gamma_0}{\partial \tau} / \frac{\partial \Gamma_0}{\partial \rho} = -\frac{\rho}{2\Gamma_0}\frac{\partial \Gamma_0}{\partial \tau} \, . \tag{3.43}$$

Upon substituting (3.43) into (3.40), i.e.

$$\rho\frac{\partial}{\partial \rho} = -2\left(\frac{1}{\Gamma_0}\frac{\partial \Gamma_0}{\partial \tau}\right)^{-1}\frac{\partial}{\partial \tau} \, , \tag{3.44}$$

the boundary condition (3.35) becomes

$$\frac{\partial \delta W}{\partial \tau} - \frac{1}{\Gamma_0}\frac{\partial \Gamma_0}{\partial \tau}\delta W = -\frac{1}{\Gamma_0}\frac{\partial \Gamma_0}{\partial \tau} f \, . \qquad \diamondsuit$$

By definition the arc length τ of an isozenithal line is measured from the reference surface positively upwards. For example, τ represents the height above the ellipsoid measured along the isozenithal. This is comparable to the orthometric height concept. The difference is that orthometric heights are measured starting from the geoid and along plumb lines. For a more thorough discussion of isozenithals and the quantities $(\rho, \frac{\partial}{\partial \rho}, \frac{\partial}{\partial \tau})$, the interested reader is asked to consult, e.g., [65] Moritz 1977 or [97] Sünkel 1978.

In order to obtain, at least for the time being, a final version of the linearized Molodensky problem, the right hand side of the boundary condition (3.39) is reconsidered in view of a parameterization in terms of the arc length τ of an isozenithal line. More precisely, the linear combination of the observables used to define the right hand sides of the various boundary conditions presented so far, see (3.26), (3.30), (3.35) and (3.39), i.e. the term f, is revised:

Lemma 7 *Let $\mathbf{x} = \mathbf{x}(\tau)$ be the equation of an isozenithal line of the normal gravity field, then $f = \Delta w + \mathbf{m}^\top \mathbf{\Delta\Gamma}$, (3.27), transforms into the following expression*

$$f = \Delta w + \left(\frac{1}{\Gamma_0}\frac{\partial \Gamma_0}{\partial \tau}\right)^{-1}\Delta\Gamma' \, , \tag{3.45}$$

where the quantity $\Delta\Gamma'$ is the component of the gravity anomaly vector $\mathbf{\Delta\Gamma}$ in the downward direction of the isozenithal.

Proof. A parameterization of the isozenithal line using the arc length τ of the isozenithal as main variable is given by

$$\mathbf{x} = \mathbf{x}(\tau)\,.$$

Then the vector

$$\mathbf{e} = \frac{d\mathbf{x}}{d\tau}$$

represents a unit tangent vector of this curve. The scalar product of the vector $\mathbf{e}$ and the gravity disturbance vector $\nabla\delta W$ yields, by means of the chain rule,

$$\mathbf{e}^\top \nabla\delta W = \frac{\partial \delta W}{\partial x_i}\frac{dx_i}{d\tau} = \frac{\partial \delta W}{\partial \tau}\,. \tag{3.46}$$

According to (3.33),

$$\mathbf{m}^\top \nabla\delta W = -\frac{\partial \delta W}{\partial p_j} p_j \tag{3.47}$$

holds. Substituting (3.38) into (3.47) leads to

$$\mathbf{m}^\top \nabla\delta W = \frac{1}{2}\rho\frac{\partial \delta W}{\partial \rho}\,. \tag{3.48}$$

Following the identity given for the partial derivatives $\frac{\partial}{\partial\rho}$ and $\frac{\partial}{\partial\tau}$ in (3.44), (3.48) transforms, together with (3.46), into

$$\begin{aligned} \mathbf{m}^\top \nabla\delta W &= -\left(\frac{1}{\Gamma_0}\frac{\partial \Gamma_0}{\partial \tau}\right)^{-1}\frac{\partial \delta W}{\partial \tau} \\ &= -\left(\frac{1}{\Gamma_0}\frac{\partial \Gamma_0}{\partial \tau}\right)^{-1}\mathbf{e}^\top \nabla\delta W\,. \end{aligned} \tag{3.49}$$

From (3.49),

$$\mathbf{m} = -\left(\frac{1}{\Gamma_0}\frac{\partial \Gamma_0}{\partial \tau}\right)^{-1}\mathbf{e} \tag{3.50}$$

can be seen immediately. Hence the vector $\mathbf{m}$ is tangential to the isozenithal and its magnitude amounts to $\left(\frac{1}{\Gamma_0}\frac{\partial \Gamma_0}{\partial \tau}\right)^{-1}$. Since τ is designed positively upwards, the minus sign implies that $\mathbf{m}$ is directed downwards. Now, using (3.50), the second term $\mathbf{m}^\top \boldsymbol{\Delta\Gamma}$ of (3.27) can be specified by

$$\mathbf{m}^\top \boldsymbol{\Delta\Gamma} = -\left(\frac{1}{\Gamma_0}\frac{\partial \Gamma_0}{\partial \tau}\right)^{-1}\mathbf{e}^\top \boldsymbol{\Delta\Gamma}\,. \tag{3.51}$$

Here, the scalar product $\mathbf{e}^\top \boldsymbol{\Delta\Gamma}$ shall define a new quantity

$$\mathbf{e}^\top \boldsymbol{\Delta\Gamma} = -\Delta\Gamma'\,, \tag{3.52}$$

which constitutes the projection of the gravity anomaly vector $\boldsymbol{\Delta\Gamma}$ into the downward direction (hence the minus sign) of the isozenithal. Finally, by substituting (3.52) together with (3.51) into (3.27), the right hand side term of the boundary condition (3.26) becomes

$$f = \Delta w + \left(\frac{1}{\Gamma_0}\frac{\partial \Gamma_0}{\partial \tau}\right)^{-1}\Delta\Gamma'\,. \qquad \diamondsuit$$

Now, (3.45), instead of (3.27), is used in (3.39). Consequently, the final form of the fundamental boundary condition of the linearized Molodensky problem reads as follows

$$\frac{\partial \delta W}{\partial \tau} - \frac{1}{\Gamma_0}\frac{\partial \Gamma_0}{\partial \tau}\delta W = -\Delta\Gamma' - \frac{1}{\Gamma_0}\frac{\partial \Gamma_0}{\partial \tau}\Delta w\,. \tag{3.53}$$

It should be emphasized that the fundamental boundary condition in its representation in (3.53), which holds on the telluroid Σ, is theoretically exact (in the framework of the linear approximation) and rigorously equivalent to the preceding formulation (3.26). No further approximations are involved.

Hence, the new variant of the linearized Molodensky problem, which is based on the revised boundary condition given above, can be formally introduced as follows:

Definition 10 *The scalar-valued or fundamental linear geodetic boundary value problem implicates the following task: the data*

$$\begin{aligned} \Delta w : \Sigma &\rightarrow \mathbb{R} \\ \Delta\Gamma' : \Sigma &\rightarrow \mathbb{R} \end{aligned}$$

are given on the telluroid Σ *and to be found is a real-valued function* $\delta W(\mathbf{x}) : ext\ \Sigma \rightarrow \mathbb{R}$*, such that*

$$\Delta \delta W(\mathbf{x}) = 0\,, \quad \mathbf{x} \in ext\ \Sigma \tag{3.54}$$

$$\left(\frac{\partial \delta W}{\partial \tau} - \frac{1}{\Gamma_0}\frac{\partial \Gamma_0}{\partial \tau}\delta W\right)\bigg|_{\Sigma} = -\Delta\Gamma' - \frac{1}{\Gamma_0}\frac{\partial \Gamma_0}{\partial \tau}\Delta w \tag{3.55}$$

is satisfied under the condition that τ *is never tangential at the boundary surface* Σ.

It follows directly from (3.55) that the linearized Molodensky problem can be categorized as an *oblique derivative problem*, since the partial derivative $\frac{\partial}{\partial \tau}$ is taken along an isozenithal line and isozenithals are (apart from the spherical case) not normal to the telluroid Σ. Obviously, oblique derivative problems are considerably more difficult than a BVP implying normal derivatives. Therefore the usual procedure to obtain a sufficiently simple BVP is to adopt certain further approximation steps. This aspect is addressed in the next section.

Again, the vectorial height anomaly $\boldsymbol{\zeta}$, which relates the unknown Earth surface σ to its known approximation, i.e. to the telluroid Σ, is determined according to (3.28) after having solved the problem described by Definition 10.

Remark 7 A simplified representation for the boundary condition of the linearized Molodensky problem, which is also found in textbooks, e.g. [28] Heiskanen&Moritz 1967, is given by

$$\left(\frac{\partial \delta W}{\partial h} - \frac{1}{\Gamma_0}\frac{\partial \Gamma_0}{\partial h}\delta W\right)\bigg|_{\Sigma} = -\Delta\Gamma\,.$$

However, this form of the boundary condition already comprises certain approximations such as: (i) instead of the partial derivative $\frac{\partial}{\partial \tau}$ taken along an isozenithal, the partial derivative $\frac{\partial}{\partial h}$ taken along the normal plumb line is applied. This is practical, since as discussed before isozenithals and plumb lines are not very different. Next, (ii) $\Delta\Gamma'$, the component of the gravity anomaly vector $\boldsymbol{\Delta\Gamma}$ in the direction of the isozenithal, is replaced by the gravity anomaly $\Delta\Gamma$, defined according to (2.50). Again, this assumption has an approximative character even though the effect is practically negligible, due to the fact that $\Delta\Gamma'$ is almost equal to $\Delta\Gamma$, since the isozenithal is directed nearly vertical. Furthermore, (iii) in comparison to (3.55) the second term is missing on the right hand side. The reason is that the telluroid Σ is in fact defined by means of Definition 6 in such a way that $\Delta w = W\big|_{\sigma} - W_0\big|_{\Sigma} = 0$ is true. Hence, the corresponding term vanishes.

3-6 Spherical approximation of the vectorial free GBVP

After having elaborated on various representations of the boundary condition of the linearized Molodensky problem in the last section, the aim of this section is to finally achieve a suitable form for the boundary condition, which satisfies the practical needs. For that purpose, an isotropic normal potential is adopted and spherical coordinates are introduced within the scope of the succeeding lemma in order to further simplify (3.55):

Lemma 8 *Using the isotropic potential (2.40) as the normal potential and employing spherical coordinates* (λ, ϕ, r) *according to (2.1), reduces the boundary condition (3.55) to*

$$\left(\frac{\partial \delta W}{\partial r} + \frac{2}{r}\delta W\right)\bigg|_{\Sigma} = -\Delta\Gamma' + \frac{2}{r}\Delta w\,. \tag{3.56}$$

Proof. Equating the normal gravity vector $\mathbf{\Gamma}_0$ to the gradient of the isotropic normal field (2.40), that is to the normal gravitational acceleration vector given by (2.41), leads to

$$\mathbf{\Gamma}_0 = -\frac{GM}{\|\mathbf{x}\|^2}\begin{bmatrix} \cos\phi\cos\lambda \\ \cos\phi\sin\lambda \\ \sin\phi \end{bmatrix}\,. \tag{3.57}$$

For the magnitude of (3.57) holds

$$\Gamma_0 = \|\mathbf{\Gamma}_0\| = \frac{GM}{\|\mathbf{x}\|^2} = \frac{GM}{r^2}\,. \tag{3.58}$$

On the other hand, representing the normal gravity vector $\mathbf{\Gamma}_0$ in quasi-spherical coordinates, cf. (3.34), thereby taking (3.29) into account, results in

$$\Gamma_0 = \|\mathbf{p}\| = \frac{1}{\rho^2} \tag{3.59}$$

for the modulus of (3.57). Consequently,

$$\rho = \frac{r}{\sqrt{GM}} \tag{3.60}$$

results by comparing (3.58) and (3.59). Thus, ρ and r differ by a constant scale factor.
As discussed before, plumb lines and isozenithals become straight lines for a homogeneous non-rotating sphere and, as a result, coincide with the spherical radii. Hence,

$$\frac{\partial}{\partial \tau} = \frac{\partial}{\partial r} \tag{3.61}$$

can be deduced for the relationship between the partial derivative taken along the isozenithal and the partial derivative with respect to the radial direction. Furthermore, the term $\frac{1}{\Gamma_0}\frac{\partial \Gamma_0}{\partial \tau}$, present on both sides of the boundary condition (3.55), transforms to

$$\frac{1}{\Gamma_0}\frac{\partial \Gamma_0}{\partial \tau} = \left(-\frac{r^2}{GM}\right)\frac{\partial}{\partial r}\left(-\frac{GM}{r^2}\right) = -2\frac{r^2}{GM}\frac{GM}{r^3} = -\frac{2}{r}\,. \tag{3.62}$$

Finally, using (3.61) and (3.62) in (3.55) yields

$$\left.\left(\frac{\partial \delta W}{\partial r} + \frac{2}{r}\delta W\right)\right|_{\Sigma} = -\Delta\Gamma' + \frac{2}{r}\Delta w\,. \qquad \diamondsuit$$

The problem to solve Laplace's equation $\Delta\delta W = 0$ outside the telluroid Σ, which is considered to be a known surface, with the above boundary condition (3.56) is generally referred to in literature as the *simple Molodensky problem*. However, before formally introducing the simple Molodensky problem in the context of a new definition, a further simplification is considered next.

As already indicated at the end of Remark 7, the surface Σ, approximating the Earth's surface σ, has been introduced at the beginning of this chapter in a quite specific manner. According to Definition 6, the Marussi telluroid Σ results from the requirement $\Delta w = W\big|_{\sigma} - W_0\big|_{\Sigma} = 0$. As a result, specification of the telluroid in such a way allows for an additional modification of the boundary condition representation (3.56). Hence, the following theorem is formulated:

Theorem 2 *The boundary condition of the simple Molodensky problem becomes*

$$\left.\left(\frac{\partial \delta W}{\partial r} + \frac{2}{r}\delta W\right)\right|_{\Sigma} = -\Delta\Gamma\,, \tag{3.63}$$

if the points Q establishing the telluroid Σ are selected such that $\Delta w = 0$.

Proof. From (3.4) and (2.46), it follows directly that the gravity potential anomaly equals zero

$$\Delta w = 0\,. \tag{3.64}$$

As discussed in the framework of Remark 7 at the end of the last section, the identity

$$\Delta\Gamma' = \Delta\Gamma \tag{3.65}$$

represents a permissible approximation. Finally, using (3.64) together with (3.65) in (3.56), the boundary condition of the simple Molodensky problem changes to

$$\left.\left(\frac{\partial \delta W}{\partial r} + \frac{2}{r}\delta W\right)\right|_{\Sigma} = -\Delta\Gamma\,. \qquad \diamondsuit$$

Remark 8 This representation of the boundary condition holds on the telluroid and is considered in nearly all practical solutions of the GBVP. It is further worth mentioning that in the context of assuming an isotropic normal potential, only the boundary operator has been approximated. That is, the influence of geometrical flattening, centrifugal potential and J_2 is neglected in the coefficients of the boundary condition (3.55) so that the spherically approximated relations of Lemma 8 or Theorem 2 are formally obtained. The telluroid Σ and the corresponding boundary data $\Delta\Gamma$ remain unaltered, still referring to the Somigliana-Pizzetti type of gravity potential of an equipotential ellipsoid. In conclusion, the described spherical approximation of the boundary condition seems feasible since it only relates quantities of the anomalous gravity field and therefore an error to the order of the flattening (0.3%, difference of the semi-major and the semi-minor axis of the reference ellipsoid divided by the radius of the reference sphere) is tolerable, cf. [66] Moritz 1980.

Thus, on the basis of the boundary condition given according to Theorem 2, the following BVP, from now on also designated as simple Molodensky's problem, can be defined at last:

Definition 11 *The geodetic boundary value problem, in spherical approximation, stands for the following problem: the data, i.e. gravity anomalies*

$$\Delta\Gamma : \Sigma \to \mathbb{R}$$

are given on the telluroid Σ and needed to be found is a real function $\delta W(\mathbf{x}) : ext\ \Sigma \to \mathbb{R}$, such that

$$\Delta\delta W(\mathbf{x}) = 0\ , \quad \mathbf{x} \in ext\ \Sigma \tag{3.66}$$

$$\left.\left(\frac{\partial \delta W}{\partial r} + \frac{2}{r}\delta W\right)\right|_{\Sigma} = -\Delta\Gamma \tag{3.67}$$

is satisfied.

Assuming for the moment that the GBVP as given above has already been solved, i.e. $\delta W(\mathbf{x})$ has been determined, then the only task left is to obtain the Earth's surface. To this end, the vectorial height anomaly $\boldsymbol{\zeta}$, which relates σ to its approximation Σ, is obtained according to (3.28). Alternatively, similar approximations as discussed in the previous sections can be introduced to simplify (3.28).

3-7 Constant radius approximation of the vectorial free GBVP

At last, this paragraph aims at finding a representation of the GBVP, which can be utilized to finally achieve a numerical solution. For that purpose, the simple Molodensky problem, see Definition 11, has to be elaborated on further. More precisely, application of the so-called *constant radius approximation* step is considered next. Constant radius approximation can be regarded simply as a one-to-one *mapping* of the given boundary values $\Delta\Gamma$ from the telluroid Σ onto the sphere S.

Consequently, the BVP (3.66)-(3.67) transforms into a new BVP defined as follows:

Definition 12 *The geodetic boundary value problem, in spherical and constant radius approximation, addresses the following problem: gravity anomalies*

$$\Delta\Gamma : S \to \mathbb{R}$$

are now given on the sphere S and to be found is a real function $\delta W(\mathbf{x}) : ext\ S \to \mathbb{R}$, such that

$$\Delta\delta W(\mathbf{x}) = 0\ , \quad \mathbf{x} \in ext\ S \tag{3.68}$$

$$\left.\left(\frac{\partial \delta W}{\partial r} + \frac{2}{R}\delta W\right)\right|_{S} = -\Delta\Gamma \tag{3.69}$$

holds.

The underlying procedure of constant radius approximation can be interpreted geometrically as follows. In the first instance, a point P given at the telluroid Σ in terms of geodetic coordinates $(\bar{h}, \bar{\phi}, \bar{\lambda})$, is replaced by a point P' given in spherical coordinates (r, ϕ, λ), see also Fig. 3.2. The two coordinate sets are related to each other by means of

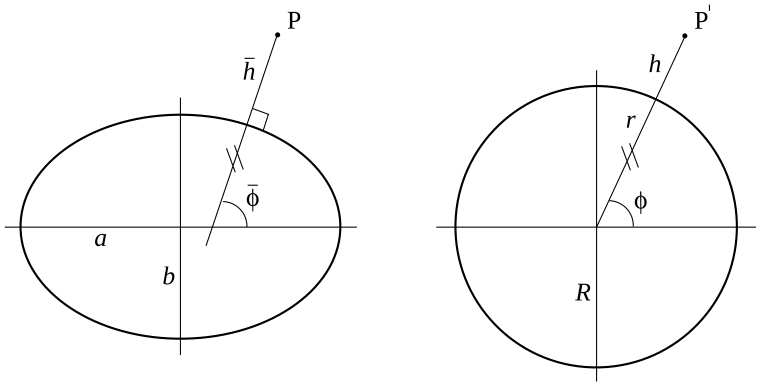


Figure 3.2: Constant radius approximation as a spherical mapping.

$$r = R + h \quad ; \quad h = \bar{h} \quad ; \quad \phi = \bar{\phi} \quad ; \quad \lambda = \bar{\lambda}\,. \tag{3.70}$$

Thus, according to (3.70), it applies that the spherical coordinates (ϕ, λ) of P' simply equal the geodetic coordinates $(\bar{\phi}, \bar{\lambda})$ of P and that the height h of P' above the sphere is made equal to the height $\bar{h}$ of P above the ellipsoid. Hence, the approximation consists of calculating with P' formally as if it were P.

In addition, based on the following reformulation of the polar coordinate r

$$r = R + h = R\left(1 + \frac{h}{R}\right)\,,$$

it follows that r differs from R by only a factor of about

$$\frac{h}{R} \cong \frac{8\,\text{km}}{6371\,\text{km}} \cong 0.001\,. \tag{3.71}$$

In fact, since an approximation error of 0.3% has already been tolerated by using spherical instead of geodetic coordinates, see Remark 9, the additional error of 0.1% when assuming a topography-free Earth, i.e. approximating r by R, can be neglected. This modification is generally also referred to as *planar approximation*.

Consequently, (3.70) becomes

$$r = R \quad ; \quad \phi = \bar{\phi} \quad ; \quad \lambda = \bar{\lambda}\,. \tag{3.72}$$

Altogether, the name *constant radius approximation* is used when the simplified problem (3.68) and (3.69) is treated instead of the BVP specified by (3.66) and (3.67).

Remark 9 Depending on the choice of the radius R of the reference sphere, the true ellipsoidal height $\bar{h}$ can differ from the true spherical height h for P by as much as 20 km. Usually, the radius R is set equal to the radius of a mean sphere, which is suitable to best approximate the ellipsoid. In that case, R is related to the ellipsoidal axes a, b by

$$R = \sqrt[3]{a^2 b}\,. \tag{3.73}$$

It is worth mentioning that a sphere defined in compliance with (3.73) lies partly above the telluroid Σ, which seems in contrast to the requirement that Laplace's equation, cf. (3.68), holds true everywhere outside the telluroid. But, since the boundary data are projected at this stage only formally from the telluroid Σ to an *arbitrary* sphere S, (3.68) is valid and no additional error is introduced here. Note, the sphere S could also theoretically be chosen to lie completely inside the telluroid.

However, the situation changes when comparing the boundary conditions (3.67) and (3.69). A properly defined telluroid, such as the *Marussi*-telluroid, deviates from the Earth's surface only by about 100 m, e.g., [19] Heck 1988. Hence, the ellipsoid adopted as a reference to represent the telluroid features the same flattening as an equipotential ellipsoid approximating the figure of the Earth. Since the transition from geodetic to spherical coordinates is equivalent to the substitution of the reference ellipsoid by a reference sphere, Fig. 3.2, the corresponding error level for the representation of the telluroid boundary surface in terms of spherical instead of geodetic coordinates is, as seen before, in the range of 0.3%. Again, this is only permissible for quantities and equations related to the anomalous field such as the boundary conditions (3.67) and (3.69).

The reason for mapping the boundary data from the telluroid to the sphere is that the sphere, as opposed to the telluroid, represents an appropriate, i.e. mathematically simple, boundary surface. This is important to solve the BVP given in Definition 12 for the disturbing gravity potential $\delta W(\mathbf{x})$ by means of a spherical harmonic analysis procedure.

Thus, completion of the physical part of Molodensky's problem provides the solution for the disturbing potential δW. The geometrical part, i.e. determination of the Earth's surface σ, follows from δW at last. Note that the sphere S, considered above within the scope of the constant radius approximation, serves only as an auxiliary surface. It provides a mathematical reference for the boundary data in view of the intended numerical examination of the problem. The approximation to the unknown surface σ remains the reference surface Σ, i.e. the Marussi telluroid specified in Definition 6. That means the vectorial height anomaly $\boldsymbol{\zeta}$, to correct for the discrepancy between σ and Σ, is still obtained according to (3.28). Again, it is worth mentioning that simplifications related to the same degree of approximation are still possible.

Remark 10 Instead of simply mapping the boundary values from the telluroid onto the sphere, two alternatives shall shortly be addressed here. Firstly, explicit analytical data continuation (ADC) strategies to generate boundary values on a spherically shaped boundary surface can be applied, see e.g. [66], [67] Moritz 1980,1990 or [88] Seitz 1997. This implies, stability provided, that ADC techniques perform with a relative error of below 0.3%. This is feasible as is shown in the quoted work of K. Seitz. Secondly, instead of using a sphere as mathematical reference surface for the harmonic analysis, the ellipsoid can be considered as a computation surface. As mentioned before, the telluroid is close to the surface of the Earth and therefore exhibits the same flattening. Thus, the ellipsoid represents a better approximation than the sphere and its use as a boundary surface theoretically eliminates the 0.3% approximation error, which arose from the spherical mapping. In order to obtain boundary values on the ellipsoid, it is sufficient (keeping in mind that the boundary data is related to the anomalous field and is therefore considered small of first order) to assume the data to be given simply at the ellipsoid, i.e. to apply *ellipsoidal mapping* so to speak. According to (3.71), this induces an approximation error of about 0.1%. However, a more sophisticated approach would be to actually compute the boundary data for the ellipsoid by means of ADC, see again [88] Seitz 1997. Note, next to the numerical solution discussed for the problem according to Definition 12, see Section 2-4 for the analytical solution of the problem, i.e. by means of Stokes integral.
At last, a comment should be made on the resemblance of the alternatives quoted here and the classical solution of the problem (3.68) and (3.69) in terms of the Molodensky's series, see e.g. [66] Moritz 1980. In fact, putting more effort into the generation of the boundary surface and the corresponding boundary data is equivalent to accounting for higher order terms in the Molodensky's series.

Chapter 4

A boundary value approach in gravity space

In the late 1970s F. Sansò pioneered a brand-new methodology for solving the vectorial free GBVP as specified in Definition 5. He transformed the overall problem, defined in the *ordinary space*, into a *fixed* BVP given in what is known as the *gravity space*. A wealth of literature is devoted to this so-called *gravity space approach*, see e.g. [77],[78],[79],[80] Sansò 1976-1978. Moreover, valuable secondary references are e.g. [65],[66] Moritz 1977,1980, providing an excellent overview of the subject. With his gravity space approach, F. Sansò managed to reduce the mathematical complexity of the problem considerably. This will be recapitulated starting from Section 4-3. Prior to this principal part, Section 4-1 elaborates the benefit of a gravity space formulation in general and Section 4-2 provides a modified definition of the vectorial free GBVP appropriate for an investigation in gravity space.

4-1 Ordinary space versus gravity space

To get started with the theory of treating the GBVP in gravity space, first of all, what has to be understood under the term *gravity space* shall be clarified. According to Section 3-3, the magnitude and the direction of the actual gravity vector $\mathbf{\Gamma} = [\Gamma_i]$, together with the true gravity potential W, have to be given throughout the surface of the Earth σ in order to solve the vectorial free GBVP. Indeed, the required knowledge of $\mathbf{\Gamma}$ gave rise to the idea to use its three Cartesian components $(\Gamma_1, \Gamma_2, \Gamma_3)$ as new curvilinear coordinates, instead of using the Cartesian coordinates (x_1, x_2, x_3) themselves. Attention is called to the fact that a similar modus operandi has already been adopted in Definition 8 and Definition 9, where the force vector of the Earth's normal gravity field has been used to define quasi spherical coordinates. These new coordinates Γ_i can be seen in two ways. On the one hand, they form curvilinear coordinates in ordinary space, and on the other hand, they can be considered as Cartesian coordinates in an auxiliary space, referred to as *gravity space*. More precisely, the gravity vector $\mathbf{\Gamma}$ forms the position vector in gravity space and its components Γ_i represent rectangular coordinates of the point at which they are computed. This is illustrated in Fig. 4.1.

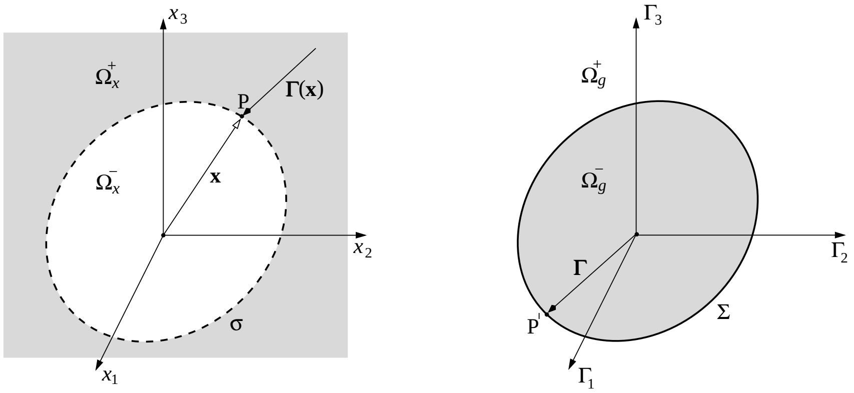


Figure 4.1: Ordinary space versus gravity space.

Moreover, the magnitude Γ of the gravity vector $\mathbf{\Gamma}$ corresponds to the length of the radius vector, when the new position vector $\mathbf{\Gamma}$ is formulated in terms of polar coordinates, cf. Definition 9. The benefit of defining such *gravity space coordinates* is that the three coordinates $\Gamma_1, \Gamma_2, \Gamma_3$ of each point of the surface σ are known. That is σ is a known surface if expressed in terms of coordinates Γ_i or, in other words, the unknown surface σ in ordinary space becomes a known surface Σ in gravity space, see again Fig. 4.1.

To guarantee a unique transition between ordinary space and gravity space, that is to obtain a one-to-one correspondence between Cartesian coordinates x_i and gravity space coordinates Γ_i, the Earth has to be considered to be non-rotating or the effect of Earth rotation has to be corrected for in advance. Moreover, the idealized assumptions for the Earth system as given in Section 3-2 apply. The reason for the postulate of a non-rotation Earth is visualized in Fig. 4.2. Outside σ, gravity diminishes along an arbitrary radius vector in the equatorial plane. This is due to the fact that with increasing distance from the geocenter, the centrifugal force increases as well. At a certain spatial point gravitational acceleration $\mathbf{g}$ is balanced by centrifugal acceleration $\mathbf{a}_z$ and gravity becomes zero. Still further away from the geocenter, gravity increases again, since the centrifugal force becomes dominant. Consequently, if a different radius vector in the equatorial plane is considered, then for another point along this direction gravity becomes zero. In fact, it is comprehensible that a closed curve C in the equatorial plane can be found where gravity is always zero. This violates the one-to-one relationship between gravity and position vector. However, these considerations are rather of theoretical nature, because from a practical point of view the region of interest lies in the vicinity near the Earth's surface. Nevertheless, in the sequel, the gravitational potential V and the gravitational acceleration vector $\mathbf{g}$ shall be considered instead of the gravity potential W and the gravity vector $\mathbf{\Gamma}$.

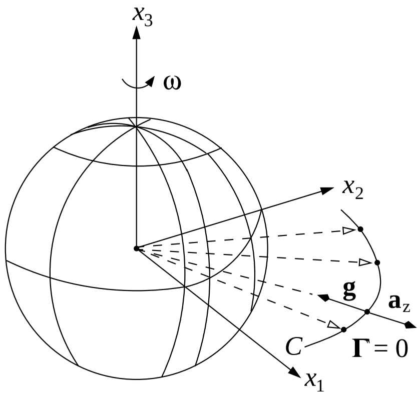


Figure 4.2: Compensation of gravitational and centrifugal acceleration.

Finally, as indicated in Fig. 4.1, the BVP in ordinary space resembles an *exterior* problem. This means that all masses generating the gravitational potential V are located in the inner region Ω_x^- confined by the surface σ. In the outer domain Ω_x^+, which is the region of interest, Laplace's equation is satisfied and the potential V is harmonic (2.30). Now, the exterior BVP in ordinary space corresponds in a one-to-one relationship to an *interior* problem of gravity space. That is, the sources generating the potential field can be imagined to be positioned in the exterior domain Ω_g^+. Consequently, the potential induced inside Ω_g^- and on the boundary surface Σ has to be considered. Thus, Ω_g^- becomes the region of interest. The circumstance that the problems in ordinary and gravity space are, in a manner of speaking, complementary to one another is directly related to the property of gravitation to decrease in magnitude with increasing distance from the geocenter. Hence, infinity in ordinary space corresponds to the origin in gravity space. It should already be stressed, that the gravitational potential tends toward zero at spatial infinity, a behaviour that will be of importance and gives reason to the considerations with respect to the regularity of the approach at the end of this chapter.

4-2 Reformulation of the vectorial free GBVP in geometry space

In the context of dealing with the vectorial free GBVP, cf. Definition 5, in gravity space, it is necessary as discussed in Section 4-1 to correct the boundary data for the influence of Earth's rotation. Hence, the resulting problem can be specified as:

Definition 13 *The rotational free version of the geodetic boundary value problem is considered as the following problem: the data are gravitational potential values and gravitational acceleration vectors given at the unknown surface* σ

$$v : \sigma \rightarrow \mathbb{R}$$
$$\tilde{\mathbf{g}} : \sigma \rightarrow \mathbb{R}^3$$

and to be found are: (i) a real function $V(\mathbf{x}) : ext\ \sigma \rightarrow \mathbb{R}$*, which is related to the given quantities by*

$$\Delta V(\mathbf{x}) = 0\,, \quad \mathbf{x} \in ext\ \sigma \tag{4.1}$$
$$\lim_{\|\mathbf{x}\|\rightarrow\infty} V(\mathbf{x}) = 0 \tag{4.2}$$
$$V\big|_\sigma = w - \frac{1}{2}\omega^2(x_1^2 + x_2^2)\big|_\sigma = v \tag{4.3}$$
$$\nabla V\big|_\sigma = \tilde{\mathbf{\Gamma}} - \omega^2\,[x_1, x_2, 0]^\top\,\big|_\sigma = \tilde{\mathbf{g}}\,, \tag{4.4}$$

and (ii) the boundary surface σ*, i.e. the physical Earth surface.*

As before, the total number of unknown quantities is four: the (external) terrestrial gravitational potential $V(\mathbf{x}) : ext\ \sigma \rightarrow \mathbb{R}$ and the components of the three-dimensional topography vector $\mathbf{x}\big|_\sigma$. In the case under consideration, i.e. determining gravitational instead of gravity potential, the four required quantities to counterbalance the unknowns cannot be measured directly but have to be deduced from the actual observations $(w, \tilde{\mathbf{\Gamma}}(\Gamma, \Phi, \Lambda))$. On this account, the effect of Earth rotation has to be reduced according to (4.3) and (4.4) and from now on, to simplify matters, the Earth is considered non-rotating, that is $\omega = 0$.

Remark 11 To treat a *free* BVP on the one hand and to correct the gravity vector for the influence of Earth rotation on the other hand, seems mutually contradictory since it requires the Earth's surface to be known. However, the effect of Earth rotation can be accounted for with sufficient accuracy on the basis of the already existing knowledge about the Earth's surface.

The forthcoming discussions on F. Sansò's gravity space approach are organized into the following sections. The next section lays the cornerstone of the new theory, i.e. the transformation from ordinary to gravity space is introduced. In this context the idea of a new potential, referred to as *adjoint potential*, is established and its relationship to the conventional potential is elaborated. In Section 4-4, the field equation for the adjoint potential, which corresponds to Laplace's equation, is derived. Furthermore, replacing the GBVP in ordinary space, the resulting *nonlinear fixed* BVP in gravity space is given. Thereafter, in Section 4-5 the aspect of linearizing the nonlinear problem is treated. A final section concerns the asymptotic relations of all involved quantities.

4-3 F. Sansò's gravity space transformation

F. Sansò's outstanding achievement to transform the vectorial free GBVP, given in its representation without the influence of Earth rotation according to Definition 13, into a problem with a fixed boundary, see e.g. [79] Sansò 1977, is based on the aforementioned idea to treat the problem in an auxiliary space, namely in gravity space. The underlying theory is composed in this section. Based on the following two definitions, a closer investigation of the relationship between the Cartesian coordinates $\mathbf{x} = [x_i]$ in ordinary space and the related coordinates in gravity space, denoted by $\boldsymbol{\xi} = [\xi_i]$ from now on, is conducted. Prior to a further definition, which introduces the new *adjoint potential* as the protagonist of gravity space theory, [85] Sansò 1981, a first lemma states that not only the gravity space coordinates $\boldsymbol{\xi}$ are of gradient type, but the Cartesian coordinates $\mathbf{x}$ as well. Thus, a mutual symmetry of ordinary and gravity space quantities becomes apparent, which in the course of a final theorem is utilized to acquire the inverse relation of the adjoint potential.

To begin with, the following transformation defining the access to gravity space is considered:

Definition 14 *Formally, gravity space is regarded as the image of the Earth's exterior domain, i.e.* $\mathbf{x} \in ext\ \sigma$*, under the following mapping*

$$\boldsymbol{\xi} = [\xi_i] := \mathbf{g}(\mathbf{x}) = [g_i] \qquad i = 1, 2, 3\,. \tag{4.5}$$

The new spatial vector $\boldsymbol{\xi}$ *represents a vector of independent Cartesian coordinates in gravity space. These coordinates* ξ_i *will be referred to as gravity space coordinates.*

Of particular interest is the restriction of the position vector in ordinary space on the right hand side of (4.5) onto the surface of the Earth, i.e. $\mathbf{x}\big|_\sigma$:

Definition 15 *The position vector* $\boldsymbol{\xi}\big|_\Sigma$ *is the image of the Earth's surface* σ *under the gravity space mapping (4.5)*

$$\boldsymbol{\xi}\big|_\Sigma = \tilde{\mathbf{g}} = \mathbf{g}(\mathbf{x}\big|_\sigma) \tag{4.6}$$

and describes a known surface in gravity space, which is denoted by Σ.

In fact, it will turn out that the surface Σ characterized above will serve as the fundamental boundary surface in gravity space. Therefore, Σ in (4.6) should not be confused with the boundary surface in ordinary space introduced in Definition 6, i.e. the ordinary telluroid surface, which has so far also been labeled by Σ. Henceforth, if not stated otherwise, Σ represents the boundary surface according to Definition 15.

Remark 12 Moreover, it should be clarified that on the one hand ordinary space is usually understood as the complete geometry space in terms of x_1, x_2, x_3 coordinates, hence comprising both the region inside and outside Earth. On the other hand, gravity space is referred to as the image of the outer Earth domain only, since the coordinate mapping could at the same time not be bijective for the regions inside and outside the Earth. Besides that, simply a lack of availability of accessible gravity information inside the Earth must be accounted for. Thus, in accordance with Definition 14 and in view of Fig. 4.1, the region Ω_g^-, i.e. the interior domain with respect to the boundary surface Σ, can be identified as gravity space.

Now, from (4.5) follows that the gravity space coordinates ξ_i represent nothing but the components g_i of the gravitational vector $\mathbf{g}$. Consequently, in view of $\mathbf{g} = \mathbf{g}(\mathbf{x})$, the new rectangular coordinates (ξ_1, ξ_2, ξ_3) themselves can be understood as functions of the old rectangular coordinates (x_1, x_2, x_3), i.e.

$$\boldsymbol{\xi} = \boldsymbol{\xi}(\mathbf{x})\,. \tag{4.7}$$

Thus, (4.7) formally constitutes a coordinate transformation (see also Section 2-1.2). It can be considered as a *mapping* of a position vector $\mathbf{x}$ of ordinary space into a position vector $\boldsymbol{\xi}$ of gravity space. Now, following the thoughts of F. Sansò, e.g. [85] Sansò 1981, a scrutinizing glance directly reveals a first important property of (4.7). That is, with (4.5) and (2.27), (4.7) is apparently of gradient type

$$\boldsymbol{\xi} = \nabla V(\mathbf{x})\,. \tag{4.8}$$

Next, having identified (4.7) as a coordinate transformation, the question to be subsequently settled is what the conditions are under which the inverse transformation

$$\mathbf{x} = \mathbf{x}(\boldsymbol{\xi}) \tag{4.9}$$

exists. Taking also Section 4-1 into account, it can be claimed that existence of a unique inverse transformation is granted under the assumption that the determinant of the Jacobian matrix, cf. (2.13), which is related to the transformation (4.7), is nonzero

$$\left|\left[\frac{\partial \xi_i}{\partial x_j}\right]\right| \neq 0\,. \tag{4.10}$$

Then, together with (4.8), (4.10) transforms into the so-called *Marussi condition*

$$\left|\left[\frac{\partial^2 V}{\partial x_i \partial x_j}\right]\right| \neq 0\,, \tag{4.11}$$

which must be satisfied accordingly. Thus, the matrix

$$\mathbf{V} = \left[\frac{\partial^2 V}{\partial x_i \partial x_j}\right]\,, \tag{4.12}$$

which has already be introduced in (2.28), constitutes the Jacobian matrix of the transformation (4.7). Note that the Marussi condition (4.11) at the same time assures invertability of $\mathbf{V}$. Hence, $\omega = 0$ and the Marussi condition set the stage for the one-to-one relationship of ordinary and gravity space.

Besides the fact that, according to (4.8), $\boldsymbol{\xi}$ is of gradient type, another fundamental property is that the vector field $\mathbf{g}(\mathbf{x})$, and thus also $\boldsymbol{\xi}(\mathbf{x})$, is a solenoidal (divergence-free) field in the region of interest. The reason for the solenoidality is the absence of mass in the Earth's exterior as agreed upon in Section 3-2. That is, in Ω_x^+ and accordingly in Ω_g^-, cf. Fig. 4.1, the vector field $\boldsymbol{\xi}(\mathbf{x})$ exhibits zero divergence

$$\nabla \cdot \boldsymbol{\xi}(\mathbf{x}) = \frac{\partial}{\partial x_i} \xi_i(\mathbf{x}) = 0 \,. \tag{4.13}$$

Indeed, together with (4.8), (4.13) expresses nothing but the fact that the gravitational potential V satisfies Laplace's equation (2.30)

$$\nabla \cdot \boldsymbol{\xi} = \nabla \cdot \nabla V = \Delta V = 0 \,, \tag{4.14}$$

which, as seen before in (2.31), can also be written in terms of the matrix $\mathbf{V}$ as follows

$$\Delta V = \operatorname{tr} \mathbf{V} = 0 \,.$$

Remark 13 Picking up the idea of Section 4-1, with the vector $\boldsymbol{\xi}$ known throughout the Earth's surface on the one hand, and the knowledge that the transition from coordinates x_i to ξ_i is justified on the other hand, the potential V might be defined as

$$V = V(\boldsymbol{\xi}) \,.$$

As already mentioned, this would alter the *free* BVP into a *fixed* BVP. In fact, the Molodensky problem of ordinary space would transform into a Dirichlet problem (Section 2-3.2) in gravity space. Yet, this approach is merely of theoretical significance as can be understood from the following reasoning. The potential V as a function of the position vector $\mathbf{x}$ satisfies Laplace's equation $V(\mathbf{x}) = 0$. According to [66] Moritz 1980, V as a function of $\boldsymbol{\xi}$ satisfies a linear partial differential equation of second order as well, which is different from Laplace's equation due to the change in coordinates. However, the coefficients of this differential equation remain practically indeterminable, since only values for $\boldsymbol{\xi}$ are known from surface measurements, cf. (4.5), but the transformation (4.7), or rather (4.8) itself, remains unknown since V is indeed the sought-after quantity. As will be seen soon, this can be remedied if not only the coordinates, but also the potential is transformed.

Next, having investigated the forward transformation from ordinary to gravity space up to now, the backward transformation (4.9)

$$\mathbf{x} = \mathbf{x}(\boldsymbol{\xi})$$

from gravity to ordinary space is considered in the following:

Lemma 9 *Comparable to (4.8), $\mathbf{x}(\boldsymbol{\xi})$ is of gradient type. That is, a function $\psi(\boldsymbol{\xi})$ exists such that*

$$\mathbf{x}(\boldsymbol{\xi}) = [x_i] = \nabla_\xi \psi(\boldsymbol{\xi}) = \left[\frac{\partial \psi}{\partial \xi_i}\right] \tag{4.15}$$

holds.

Proof. A first glance on the Jacobian matrix $\left[\frac{\partial x_i}{\partial \xi_j}\right]$ of the inverse transformation (4.9) yields the following relation, by means of Lemma 1 together with (4.8) and (4.12),

$$\left[\frac{\partial x_i}{\partial \xi_j}\right] = \left[\frac{\partial \xi_j}{\partial x_k}\right]^{-1} = \mathbf{V}^{-1} \,. \tag{4.16}$$

Once more, existence of $\mathbf{V}^{-1}$ follows from the Marussi condition (4.11). And since $\mathbf{V}$ is a symmetric matrix, so is $\mathbf{V}^{-1}$. Consequently, the equality

$$\frac{\partial x_i}{\partial \xi_j} = \frac{\partial x_j}{\partial \xi_i} \tag{4.17}$$

is true. Taking (2.21) into consideration, (4.17) directly implies

$$\nabla \times \mathbf{x}(\boldsymbol{\xi}) = \mathbf{0} \,, \tag{4.18}$$

that means $\mathbf{x}(\boldsymbol{\xi})$ is a vector field with zero curl at every point of the region of interest. According to, e.g., [50] Kellogg 1967, vanishing of the curl at every point of a simply connected region is a necessary and sufficient condition that the vector field is said to be *irrotational*, i.e. free from vortices. It is well known in vector calculus that irrotational vector fields are characterized by the fact that they have a potential function, e.g. [50] KELLOGG 1967, [8] BURG 1990. Hence, the irrotational vector field $\mathbf{x}(\boldsymbol{\xi})$ can indeed be interpreted as the gradient field of a scalar function denoted, e.g., by $\psi(\boldsymbol{\xi})$

$$\mathbf{x}(\boldsymbol{\xi}) = \nabla_\xi \psi(\boldsymbol{\xi}) \,. \qquad \diamondsuit$$

Thus, $\psi(\boldsymbol{\xi})$ constitutes a scalar potential function in gravity space, which will be explicitely introduced in a separate definition subsequent to the remark given next.

Remark 14 It is worthwhile to call attention to the following conclusion

$$\nabla \times \mathbf{x}(\boldsymbol{\xi}) = \mathbf{0} \quad \Rightarrow \quad \exists\ \psi(\boldsymbol{\xi}) : \mathbf{x}(\boldsymbol{\xi}) = \nabla_\xi \psi(\boldsymbol{\xi}) \,,$$

which, in short, results from the line of argument given above. To prove the inverse conclusion

$$\mathbf{x}(\boldsymbol{\xi}) = \nabla_\xi \psi(\boldsymbol{\xi}) \quad \Rightarrow \quad \nabla \times \mathbf{x}(\boldsymbol{\xi}) = \mathbf{0}$$

is trivial, since the curl of the gradient field of any scalar function is always zero

$$\nabla \times \nabla_\xi \psi(\boldsymbol{\xi}) = \mathbf{0} \,,$$

which can be easily verified by direct calculation. Note the fact that the gradient field of the potential function $\psi(\boldsymbol{\xi})$ is irrotational analogous to (2.32).

It should be emphasized that according to (4.8) and (4.15), there exists mutual symmetry between $\mathbf{x}$ and V on the one hand and ψ and $\boldsymbol{\xi}$ on the other hand. Due to this striking symmetry it will be possible to express one potential in terms of the other, which is the subject addressed in the following. Thus, the question that finally will be settled is the problem of explicitly finding the scalar function $\psi(\boldsymbol{\xi})$, which in fact represents a new potential defined in gravity space.

Definition 16 *The potential $\psi(\boldsymbol{\xi})$ of gravity space, which is in agreement with Lemma 9, is given by*

$$\psi(\boldsymbol{\xi}) := \mathbf{x}^\top \nabla V(\mathbf{x}) - V(\mathbf{x}) \quad ; \quad \mathbf{x} = \mathbf{x}(\boldsymbol{\xi}) \,, \tag{4.19}$$

and is referred to from now on as adjoint potential.

Remark 15 It is worth mentioning that (4.19) constitutes a relation of particular interest. That is to say, the transformation of the gravitational potential $V(\mathbf{x})$, defined in ordinary space, into the adjoint potential $\psi(\boldsymbol{\xi})$, given in gravity space. By definition, the adjoint potential ψ is a function of the independent coordinate vector $\boldsymbol{\xi}$. The expression on the right hand side of (4.19) must therefore represent a function of $\boldsymbol{\xi}$ as well. Thus, the coordinate vector $\mathbf{x}$ in (4.19) must also be understood as a function of $\boldsymbol{\xi}$, i.e. $\mathbf{x} = \mathbf{x}(\boldsymbol{\xi})$. In fact, this reasoning is closely related to Lemma 9, as can be shown by means of simple computations. That is, differentiating $\psi(\boldsymbol{\xi})$ according to (4.19) with respect to $\boldsymbol{\xi}$, and taking $\boldsymbol{\xi} = \nabla V(\mathbf{x})$, (4.8), into account yields

$$\frac{\partial \psi}{\partial \xi_i} = \frac{\partial x_k}{\partial \xi_i} \frac{\partial V}{\partial x_k} + x_k \frac{\partial \xi_k}{\partial \xi_i} - \frac{\partial V}{\partial x_k} \frac{\partial x_k}{\partial \xi_i} = x_i \,.$$

As a matter of fact, this derivation represents nothing else but the relation given by Lemma 9. On the one hand, it confirms the aforementioned conclusion, namely that $\mathbf{x}$ in (4.19) is indeed a function of $\boldsymbol{\xi}$. On the other hand, it is demonstrated that the adjoint potential $\psi(\boldsymbol{\xi})$, as specified in Definition 16, is in agreement with Lemma 9.

Remark 16 On identifying

$$u = V \quad , \quad \mathbf{p} = \nabla V \quad , \quad U = \psi \quad , \quad \mathbf{P} = \nabla_\xi \psi \quad , \quad \mathbf{X} = \boldsymbol{\xi} \,,$$

and by comparison with (2.88), it is understandable that the equations constituting F. Sansò's gravity space approach, i.e. (4.8), (4.15) and (4.19), substantiate nothing else but a *Legendre transformation*. Thus, they belong to the family of contact transformations. Furthermore, in accordance with the theory of Legendre transformations, as outlined in Section 2-5.3, the inverse relation of (4.19), i.e. the transformation of the adjoint potential $\psi(\boldsymbol{\xi})$ into the gravitational potential $V(\mathbf{x})$, can be directly specified due to the complete symmetry of (2.88) and (2.92).

Theorem 3 *The inverse relation of (4.19) follows in view of (2.92) and by taking the identities given in the previous remark into account,*

$$V(\mathbf{x}) = \boldsymbol{\xi}^\top \nabla_\xi \psi(\boldsymbol{\xi}) - \psi(\boldsymbol{\xi}) \quad ; \quad \boldsymbol{\xi} = \boldsymbol{\xi}(\mathbf{x})\,. \tag{4.20}$$

Proof. In order to verify Theorem 3 one possibility is to formally differentiate $V(\mathbf{x})$ with respect to $\boldsymbol{\xi}$ by applying the chain rule of differential calculus

$$\frac{\partial V}{\partial \xi_i} = \frac{\partial V}{\partial x_j}\frac{\partial x_j}{\partial \xi_i}. \tag{4.21}$$

The first fraction on the right hand side in (4.21) resembles (4.8) and the second fraction is reformulated by employing Lemma 9. Hence, (4.21) becomes

$$\frac{\partial V}{\partial \xi_i} = \xi_j \frac{\partial}{\partial \xi_i}\left(\frac{\partial \psi}{\partial \xi_j}\right) \tag{4.22}$$

and finally

$$\frac{\partial V}{\partial \xi_i} = \frac{\partial}{\partial \xi_i}\left(\xi_j \frac{\partial \psi}{\partial \xi_j}\right) - \frac{\partial \psi}{\partial \xi_i} \tag{4.23}$$

is attained. The identity of (4.22) and (4.23) can be validated by formally evaluating the first term on the right hand side of (4.23) by means of the product rule of differentiation. As a matter of fact, $\frac{\partial \psi}{\partial \xi_i}$ is canceled out and (4.22) can be obtained. Next,

$$V = \xi_j \frac{\partial \psi}{\partial \xi_j} - \psi + c \tag{4.24}$$

follows from integration. Without going into detail here, the arbitrary integration constant c can be eliminated, i.e. $c = 0$, by specifying the value of ψ to be zero at the origin in gravity space. This choice is motivated by the fact that, according to Section 4-1, the origin in gravity space corresponds to infinity in ordinary space, where the gravitational potential V vanishes as well. Consequently, (4.24) becomes

$$V(\mathbf{x}) = \boldsymbol{\xi}^\top \nabla_\xi \psi(\boldsymbol{\xi}) - \psi(\boldsymbol{\xi})\,. \qquad \diamondsuit$$

Remark 17 Reference has already been made to the underlying symmetry of ordinary and gravity space quantities. Now, this symmetry can, for example, be exploited to retrieve the adjoint potential, (4.19), as the inverse of the transformation (4.20) and thus to give evidence for the consistency of these relations. That is, solving (4.20) for $\psi(\boldsymbol{\xi})$, substituting $\boldsymbol{\xi}$ by $\nabla V(\mathbf{x})$ according to (4.8) and adopting $\mathbf{x}$ for $\nabla_\xi \psi(\boldsymbol{\xi})$ according to (4.15) eventually yields

$$\psi(\boldsymbol{\xi}) = \mathbf{x}^\top \nabla V(\mathbf{x}) - V(\mathbf{x}) \quad ; \quad \mathbf{x} = \mathbf{x}(\boldsymbol{\xi})\,.$$

It is of course feasible to start from (4.19) and to derive (4.20) by means of the symmetry properties as the inverse transformation of (4.19). In fact, this possibility is used to deduce the corresponding boundary condition of the GBVP in gravity space, which will be introduced in the next paragraph.

Remark 18 In [48] Keller 1987 an alternative way, namely by means of considerations from the point of view of projective geometry, is presented to show that it is indeed possible to classify the transformations (4.19), (4.8) and (4.15) as part of the family of contact transformations. In fact, W. Keller points out that (4.19), (4.8) and (4.15) result from the polarity on the imaginary unit circle, the simplest kind of a contact transformation. However, despite this geometrically meaningful and descriptive possibility given in [48] Keller 1987 to affirm the relation according to Definition 16 and to prove the assertion of Theorem 3, a different, in some way more straightforward, approach has been considered above.

4-4 The nonlinear GBVP in gravity space

As is generally known, the basic field equation in ordinary space, to account for the harmonic character of the potential V, is Laplace's equation, which in terms of the matrix $\mathbf{V}$ reads as follows $\Delta V = \operatorname{tr}\mathbf{V} = 0$. In the last section, by means of (4.8) and (4.14), $\mathbf{V}$ and consequently Laplace's equation as well are, in a manner of speaking, related to the forward transformation from ordinary to gravity space $\boldsymbol{\xi} = \boldsymbol{\xi}(\mathbf{x})$. In fact, $\mathbf{V}$ has been found to constitute the Jacobian matrix of this transformation. Now, the present section considers the problem in reverse. That is, starting with the adjoint potential ψ, its matrix of second order derivatives $\boldsymbol{\Psi}$ is considered, which eventually results in a partial differential equation for the adjoint potential as provided by Lemma 10 and Lemma 11. This is followed by an investigation on the associated boundary condition to come up with the nonlinear fixed GBVP in gravity space.

To get started, the matrix $\mathbf{\Psi}$ of second-order derivatives of ψ is introduced. Then, as indicated before, $\mathbf{\Psi}$ is identified as the Jacobian matrix of the inverse transformation from gravity to ordinary space $\mathbf{x} = \mathbf{x}(\boldsymbol{\xi})$. The matrix $\mathbf{\Psi}$ is then related to the matrix $\mathbf{V}$ in order to find the field equation in gravity space as accomplished in the next lemma:

Lemma 10 *The basic field equation in gravity space reads as*

$$\operatorname{tr} \mathbf{\Psi}^{-1} = 0 \, , \tag{4.25}$$

subject to the matrix $\mathbf{\Psi}$

$$\mathbf{\Psi} = [\Psi_{ij}] = \nabla_\xi \left(\nabla_\xi \psi \right) = \left[\frac{\partial^2 \psi}{\partial \xi_i \partial \xi_j} \right] \tag{4.26}$$

composed of the second order derivatives of the adjoint potential $\psi(\boldsymbol{\xi})$*.*

Proof. As a direct consequence of Lemma 9, (4.26) can be written as

$$\mathbf{\Psi} = [\Psi_{ij}] = \nabla_\xi \mathbf{x}(\boldsymbol{\xi}) = \left[\frac{\partial x_i}{\partial \xi_j} \right] . \tag{4.27}$$

In Section 4-3, $\mathbf{V}$ as a function of V, see (4.12), has been found to represent the Jacobian matrix of the transformation $\boldsymbol{\xi} = \boldsymbol{\xi}(\mathbf{x})$. Now due to symmetry reasons, it applies that $\mathbf{\Psi}$ as a function of ψ, see (4.26), constitutes the Jacobian matrix of the inverse transformation $\mathbf{x} = \mathbf{x}(\boldsymbol{\xi})$ from gravity to ordinary space. Hence, having identified $\mathbf{V}$ as the Jacobian matrix of the forward transformation, and $\mathbf{\Psi}$ as the Jacobian matrix of the backward transformation, according to Lemma 1, the following identity is true

$$\mathbf{V} = \mathbf{\Psi}^{-1} . \tag{4.28}$$

Hence, the basic field equation of gravity space can be deduced from the previous definition, (2.31), of Laplace's equation

$$\Delta V = \operatorname{tr} \mathbf{V} = \operatorname{tr} \mathbf{\Psi}^{-1} = 0 . \qquad \diamond$$

The partial differential equation for the adjoint potential according to Lemma 10 can be simplified further:

Lemma 11 *Transformation of the field equation in gravity space, (4.25), leads to*

$$(\operatorname{tr} \mathbf{\Psi})^2 - \operatorname{tr} \mathbf{\Psi}^2 = 0 . \tag{4.29}$$

Proof. The well known relation in linear algebra

$$\mathbf{\Psi}^{-1} = \frac{\operatorname{adj} \mathbf{\Psi}}{|\mathbf{\Psi}|} = \frac{1}{|\mathbf{\Psi}|} \left[\bar{\Psi}_{ij} \right] , \tag{4.30}$$

which is referred to as *Cramer's rule*, e.g. [108] Zurmühl 1961, allows computation of the inverse of the matrix $\mathbf{\Psi}$ from the corresponding adjoint matrix $\operatorname{adj} \mathbf{\Psi} = [\bar{\Psi}_{ij}]$. Hence, in order to arrive at $\operatorname{tr} \mathbf{\Psi}^{-1}$ as required in Lemma 10, the trace operator must be applied to (4.30)

$$\begin{aligned}
\operatorname{tr} \mathbf{\Psi}^{-1} &= \frac{1}{|\mathbf{\Psi}|} \operatorname{tr} \left[\bar{\Psi}_{ij} \right] \\
&= \frac{1}{|\mathbf{\Psi}|} \left(\bar{\Psi}_{11} + \bar{\Psi}_{22} + \bar{\Psi}_{33} \right) \\
&= \frac{1}{|\mathbf{\Psi}|} \left(\Psi_{22}\Psi_{33} - \Psi_{23}^2 + \Psi_{11}\Psi_{33} - \Psi_{13}^2 + \Psi_{11}\Psi_{22} - \Psi_{12}^2 \right) \\
&= \frac{-1}{2\,|\mathbf{\Psi}|} \left(\Psi_{1i}\Psi_{i1} + \Psi_{2i}\Psi_{i2} + \Psi_{3i}\Psi_{i3} - \Psi_{11}^2 - \Psi_{22}^2 - \Psi_{33}^2 - 2\left[\Psi_{11}\Psi_{22} + \Psi_{11}\Psi_{33} + \Psi_{22}\Psi_{33} \right] \right) \\
&= \frac{-1}{2\,|\mathbf{\Psi}|} \left(\Psi_{ji}\Psi_{ij} - \left[\Psi_{11} + \Psi_{22} + \Psi_{33} \right]^2 \right) \\
&= \frac{-1}{2\,|\mathbf{\Psi}|} \left(\operatorname{tr} \mathbf{\Psi}^2 - (\operatorname{tr} \mathbf{\Psi})^2 \right) .
\end{aligned} \tag{4.31}$$

Consequently, the previous field equation $\mathrm{tr}\,\boldsymbol{\Psi}^{-1} = 0$, (4.25), transforms, by means of the last relation in (4.31), into the new partial differential equation for the adjoint potential $\psi(\boldsymbol{\xi})$

$$(\mathrm{tr}\,\boldsymbol{\Psi})^2 - \mathrm{tr}\,\boldsymbol{\Psi}^2 = 0\,. \qquad \diamondsuit$$

Remark 19 According to (4.31), the field equation (4.29) may also be written in the following form

$$\Psi_{11}\Psi_{22} - \Psi_{12}^2 + \Psi_{22}\Psi_{33} - \Psi_{23}^2 + \Psi_{11}\Psi_{33} - \Psi_{13}^2 = 0\,,$$

which shows nicely that (4.29) is in fact a partial differential equation for $\psi(\boldsymbol{\xi})$ with known coefficients, all +1 or -1, but unfortunately a nonlinear one.

Having finished the considerations in view of the field equation, which the adjoint potential has to satisfy, the second aspect to be treated is related to the associated boundary condition. For this purpose, (4.19) is solved for V and restricted onto the boundary surface σ

$$V\big|_\sigma = \left(\mathbf{x}^\top \nabla V\right)\Big|_\sigma - \psi\big|_\Sigma\,. \tag{4.32}$$

Then, using (4.15) to substitute $\mathbf{x}$ and (4.8) to replace ∇V, leads to

$$\begin{aligned} V\big|_\sigma &= \left(\nabla_\xi \psi^\top \boldsymbol{\xi}\right)\Big|_\Sigma - \psi\big|_\Sigma \\ &= \left(\boldsymbol{\xi}^\top \nabla_\xi \psi - \psi\right)\Big|_\Sigma\,. \end{aligned} \tag{4.33}$$

Admittedly, the representation in (4.33) obtained for the requested boundary condition has already been established by means of Theorem 3 in Section 4-3. However, the above considerations in terms of (4.32) and (4.33) nicely re-emphasize the consistency of the presented gravity space theory, and provide an independent alternative to deduce (4.20) from (4.19) as has already been indicated in Remark 17 at the end of the last section.

Finally, it can be concluded that in the new coordinates $\boldsymbol{\xi}$ the adjoint potential ψ solves the following BVP in gravity space, which replaces the vectorial free GBVP in ordinary space according to Definition 13:

Definition 17 *The geodetic boundary value problem in gravity space is understood as the following problem: the data, i.e. gravitational potential values, are assumed to be given at the known surface* Σ

$$v : \Sigma \to \mathbb{R}$$

and to be found is a real function $\psi(\boldsymbol{\xi}) : int\ \Sigma \to \mathbb{R}$

$$(\mathrm{tr}\,\boldsymbol{\Psi})^2 - \mathrm{tr}\,\boldsymbol{\Psi}^2 = 0\,, \quad \boldsymbol{\xi} \in int\ \Sigma \tag{4.34}$$

$$\left(\boldsymbol{\xi}^\top \nabla_\xi \psi - \psi\right)\Big|_\Sigma = v\,, \tag{4.35}$$

which is the solution of the second-order partial differential equation (4.34) under the boundary condition (4.35).

Concerning Definition 17, it should be added that (4.34) and (4.35) constitute a nonlinear oblique BVP with a Monge-Ampère type of differential equation and a linear boundary condition. They establish the GBVP in gravity space, a BVP with a *fixed* boundary, since in contrast to the Earth's surface σ, the boundary surface Σ is known as recalled from Definition 15

$$\boldsymbol{\xi}\big|_\Sigma = \tilde{\mathbf{g}} = \mathbf{g}(\mathbf{x}\big|_\sigma)\,.$$

After having obtained the solution for $\psi(\boldsymbol{\xi})$, the Earth's surface σ, according to Lemma 9 and the interrelationship of σ and Σ shown in Definition 15, is derived as follows

$$\mathbf{x}\big|_\sigma = \nabla_\xi \psi(\boldsymbol{\xi}\big|_\Sigma)\,. \tag{4.36}$$

4-5 Linearization of the GBVP in gravity space

In the course of the linearization process of the vectorial free GBVP in ordinary space, Section 3-4, the telluroid surface Σ and an associated normal gravity potential W_0 were introduced to approximate the physical shape of the Earth and its true gravity potential. In this section, linearization of the fixed GBVP in gravity space is accomplished in a similar manner, only there exists a subtle distinction. That is, in contrast to the linearization procedure of the GBVP in ordinary space, the necessity to adopt a reference surface in order to approximate the true boundary surface becomes meaningless, since in gravity space the BVP represents a problem with a fixed boundary surface. This major difference of gravity and ordinary space treatment proves to be of significant benefit for the actual numerical solution of the GBVP as will be discussed in a later chapter. Still, the idea presented here to approximate the adjoint potential with an *adjoint normal potential* is closely related to the linearization procedure in ordinary space.
At the beginning, by means of a first theorem the adjoint normal potential ψ_0, which corresponds to the normal gravitational potential $V_0 = \frac{GM}{\|\mathbf{x}\|}$, is established. The matrix $\mathbf{\Psi}_0$ of second order derivatives of ψ_0 is then derived in a separate lemma. As much as ψ_0 represents a first order approximation of the true adjoint potential ψ, $\mathbf{\Psi}_0$ should be regarded as an initial approximate of $\mathbf{\Psi}$. A second lemma confirms that the adjoint normal potential ψ_0 satisfies the field equation of gravity space specified in Lemma 11 just as ψ does. For this purpose, (4.29) is explicitely evaluated by making use of the first lemma. Next, the nonlinear partial differential equation for ψ, cf. Remark 19, is linearized in a third lemma and elaborated further in a fourth. As a consequence thereof, it turns out that the *adjoint disturbing potential* satisfies a partial differential equation in gravity space other than Laplace's equation as addressed in detail in an additional remark. These considerations, concerning the field equation for the adjoint disturbing potential, are followed by investigations of the corresponding boundary condition. In particular, the concept of the *gravimetric telluroid* is introduced in a separate definition. Finally, the linearized version of the BVP (4.34)-(4.35), given at the end of the last section, concludes this section.

First of all, the adjoint normal potential ψ_0 is derived as mentioned before in the context of a new theorem:

Theorem 4 *In case for the gravitational normal potential V_0 holds*

$$V_0(\mathbf{x}) = \frac{GM}{\|\mathbf{x}\|}\,, \tag{4.37}$$

then the corresponding adjoint normal potential ψ_0 reads as

$$\psi_0 = -2\sqrt{GM\|\boldsymbol{\xi}\|}\,. \tag{4.38}$$

Proof. As a start, (4.37) is substituted into (4.8) as well as into (4.19) in order to obtain the following two equations

$$\boldsymbol{\xi} = -\frac{GM}{\|\mathbf{x}\|^2}\frac{\mathbf{x}}{\|\mathbf{x}\|} \tag{4.39}$$

$$\psi_0 = -\frac{GM}{\|\mathbf{x}\|^3}\mathbf{x}^\top\mathbf{x} - \frac{GM}{\|\mathbf{x}\|} = -2\frac{GM}{\|\mathbf{x}\|}\,. \tag{4.40}$$

From (4.39) follows immediately

$$\|\boldsymbol{\xi}\| = \frac{GM}{\|\mathbf{x}\|^2}\,. \tag{4.41}$$

Solving (4.41) for $\|\mathbf{x}\|$ and insertion of the resulting expression into (4.40) leads to

$$\psi_0 = -2\frac{GM}{\sqrt{\frac{GM}{\|\boldsymbol{\xi}\|}}} = -2\sqrt{GM\|\boldsymbol{\xi}\|}\,. \qquad \diamond$$

Setting up the Hessian matrix related to the adjoint normal potential ψ_0 is the objective of the next lemma in order to prepare for the lemma that follows thereafter:

Lemma 12 *Let $\mathbf{\Psi}_0$ represent the matrix of the second order derivatives of the adjoint normal potential ψ_0, then $\mathbf{\Psi}_0$ reads as*

$$\mathbf{\Psi}_0 = \left[\Psi^0_{ij}\right] = \left[\frac{\partial^2\psi_0}{\partial\xi_i\partial\xi_j}\right] = \frac{-\sqrt{GM}}{\|\boldsymbol{\xi}\|^{3/2}}\left[\delta_{ij} - \frac{3}{2}\frac{\xi_i\xi_j}{\|\boldsymbol{\xi}\|^2}\right]\,. \tag{4.42}$$

Proof. At first, the expression for the adjoint normal potential $\psi_0(\boldsymbol{\xi})$ given in Theorem 4 is written in the following form

$$\psi_0 = -2\sqrt{GM}\left(\boldsymbol{\xi}^\top\boldsymbol{\xi}\right)^{1/4} .$$

Next, the gradient of the adjoint normal potential $\psi_0(\boldsymbol{\xi})$ with respect to the $\boldsymbol{\xi}$ coordinates is given by

$$\nabla_\xi\psi_0 = -2\sqrt{GM}\frac{1}{4}\left(\boldsymbol{\xi}^\top\boldsymbol{\xi}\right)^{-3/4}2\boldsymbol{\xi} = \frac{-\sqrt{GM}}{\|\boldsymbol{\xi}\|^{3/2}}\boldsymbol{\xi}\,, \tag{4.43}$$

and by differentiating (4.43) once more, the required matrix of the second order partial derivatives of ψ_0 is obtained

$$\begin{aligned}\nabla_\xi(\nabla_\xi\psi_0) = \left[\Psi^0_{ij}\right] &= -\sqrt{GM}\left[\frac{1}{\|\boldsymbol{\xi}\|^{3/2}}\mathbf{I} + \frac{-3}{2\|\boldsymbol{\xi}\|^{7/2}}\left[\boldsymbol{\xi}\boldsymbol{\xi}^\top\right]\right] \\ &= \frac{-\sqrt{GM}}{\|\boldsymbol{\xi}\|^{3/2}}\left[\delta_{ij} - \frac{3}{2}\frac{\xi_i\xi_j}{\|\boldsymbol{\xi}\|^2}\right] . \qquad \diamondsuit\end{aligned}$$

It is shown next that the adjoint normal potential ψ_0 defined according to Theorem 4 fulfills the field equation in gravity space as given in Lemma 11:

Lemma 13 *The adjoint normal potential ψ_0, (4.38), satisfies the following partial differential equation*

$$(\operatorname{tr}\boldsymbol{\Psi}_0)^2 - \operatorname{tr}\boldsymbol{\Psi}_0^2 = 0\,. \tag{4.44}$$

Proof. To begin with, by taking (4.42) into account, the first term of (4.44) is evaluated

$$\begin{aligned}(\operatorname{tr}\boldsymbol{\Psi}_0)^2 = \left(\operatorname{tr}\left[\frac{-\sqrt{GM}}{\|\boldsymbol{\xi}\|^{3/2}}\left[\delta_{ij} - \frac{3}{2}\frac{\xi_i\xi_j}{\|\boldsymbol{\xi}\|^2}\right]\right]\right)^2 &= \left(\frac{-\sqrt{GM}}{\|\boldsymbol{\xi}\|^{3/2}}\left(\delta_{ii} - \frac{3}{2}\frac{\xi_i\xi_i}{\|\boldsymbol{\xi}\|^2}\right)\right)^2 \\ &= \left(\frac{-\sqrt{GM}}{\|\boldsymbol{\xi}\|^{3/2}}\left(3 - \frac{3}{2}\right)\right)^2 \\ &= \frac{9}{4}\frac{GM}{\|\boldsymbol{\xi}\|^3} .\end{aligned} \tag{4.45}$$

Consequently, the second term of (4.44) gives

$$\begin{aligned}\operatorname{tr}\boldsymbol{\Psi}_0^2 = \operatorname{tr}\left[\frac{-\sqrt{GM}}{\|\boldsymbol{\xi}\|^{3/2}}\left[\delta_{ij} - \frac{3}{2}\frac{\xi_i\xi_j}{\|\boldsymbol{\xi}\|^2}\right]\right]^2 &= \operatorname{tr}\left[\frac{GM}{\|\boldsymbol{\xi}\|^3}\left[\delta_{ij} - \frac{3}{2}\frac{\xi_i\xi_j}{\|\boldsymbol{\xi}\|^2}\right]\left[\delta_{jk} - \frac{3}{2}\frac{\xi_j\xi_k}{\|\boldsymbol{\xi}\|^2}\right]\right] \\ &= \frac{GM}{\|\boldsymbol{\xi}\|^3}\operatorname{tr}\left[\delta_{ik} - \frac{3}{2}\frac{\xi_i\xi_k}{\|\boldsymbol{\xi}\|^2} - \frac{3}{2}\frac{\xi_i\xi_k}{\|\boldsymbol{\xi}\|^2} + \frac{9}{4}\frac{\xi_i\xi_k\|\boldsymbol{\xi}\|^2}{\|\boldsymbol{\xi}\|^4}\right] \\ &= \frac{GM}{\|\boldsymbol{\xi}\|^3}\left(\delta_{ii} - \frac{3}{2}\frac{\xi_i\xi_i}{\|\boldsymbol{\xi}\|^2} - \frac{3}{2}\frac{\xi_i\xi_i}{\|\boldsymbol{\xi}\|^2} + \frac{9}{4}\frac{\xi_i\xi_i}{\|\boldsymbol{\xi}\|^2}\right) \\ &= \frac{GM}{\|\boldsymbol{\xi}\|^3}\left(3 - \frac{3}{2} - \frac{3}{2} + \frac{9}{4}\right) \\ &= \frac{9}{4}\frac{GM}{\|\boldsymbol{\xi}\|^3} .\end{aligned} \tag{4.46}$$

Therefore, by means of (4.45) and (4.46), it follows directly

$$(\operatorname{tr}\boldsymbol{\Psi}_0)^2 - \operatorname{tr}\boldsymbol{\Psi}_0^2 = \frac{9}{4}\frac{GM}{\|\boldsymbol{\xi}\|^3} - \frac{9}{4}\frac{GM}{\|\boldsymbol{\xi}\|^3} = 0\,. \qquad \diamondsuit$$

Prior to the next lemma, aiming at the linearization of the nonlinear GBVP in gravity space given at the end of Section 4-4, the following preparatory considerations are made. That is, the unknown adjoint potential $\psi(\boldsymbol{\xi})$ is split into its known normal part $\psi_0(\boldsymbol{\xi})$ and its disturbing part $\delta\psi(\boldsymbol{\xi})$

$$\psi = \psi_0 + \delta\psi\,. \tag{4.47}$$

Accordingly, proceeding in the same manner with regard to the matrix of second order derivatives $\boldsymbol{\Psi}$ yields

$$\boldsymbol{\Psi} = \boldsymbol{\Psi}_0 + \delta\boldsymbol{\Psi}\,, \tag{4.48}$$

subject to the matrix $\delta\boldsymbol{\Psi}$

$$\delta\boldsymbol{\Psi} = [\delta\Psi_{ij}] = \left[\frac{\partial^2\delta\psi}{\partial\xi_i\partial\xi_j}\right] . \tag{4.49}$$

This provides all requirements in order to address the requested linearization step as considered in the next lemma:

Lemma 14 *The nonlinear field equation (4.29) can be approximated by the following partial differential equation*

$$\mathrm{tr}\,\boldsymbol{\Psi}_0\mathrm{tr}\,\delta\boldsymbol{\Psi} - \mathrm{tr}\,[\boldsymbol{\Psi}_0\delta\boldsymbol{\Psi}] = 0\,, \tag{4.50}$$

by taking advantage of Lemma 13 and by neglecting terms of order $\mathrm{O}\big(\delta\boldsymbol{\Psi}^2\big)$.

Proof. At first, insertion of (4.48) into (4.29) produces

$$\begin{aligned} 0 = (\mathrm{tr}\,\boldsymbol{\Psi})^2 - \mathrm{tr}\,\boldsymbol{\Psi}^2 &= (\mathrm{tr}\,[\boldsymbol{\Psi}_0 + \delta\boldsymbol{\Psi}])^2 - \mathrm{tr}\,[\boldsymbol{\Psi}_0 + \delta\boldsymbol{\Psi}]^2 \\ &= (\mathrm{tr}\,\boldsymbol{\Psi}_0 + \mathrm{tr}\,\delta\boldsymbol{\Psi})^2 - \mathrm{tr}\,\left[\boldsymbol{\Psi}_0^2 + 2\boldsymbol{\Psi}_0\delta\boldsymbol{\Psi} + \delta\boldsymbol{\Psi}^2\right] \\ &= (\mathrm{tr}\,\boldsymbol{\Psi}_0)^2 + 2\mathrm{tr}\,\boldsymbol{\Psi}_0\mathrm{tr}\,\delta\boldsymbol{\Psi} + (\mathrm{tr}\,\delta\boldsymbol{\Psi})^2 - \mathrm{tr}\,\boldsymbol{\Psi}_0^2 - 2\mathrm{tr}\,[\boldsymbol{\Psi}_0\delta\boldsymbol{\Psi}] - \mathrm{tr}\,\delta\boldsymbol{\Psi}^2 \\ &= 2\mathrm{tr}\,\boldsymbol{\Psi}_0\mathrm{tr}\,\delta\boldsymbol{\Psi} - 2\mathrm{tr}\,[\boldsymbol{\Psi}_0\delta\boldsymbol{\Psi}] + (\mathrm{tr}\,\boldsymbol{\Psi}_0)^2 - \mathrm{tr}\,\boldsymbol{\Psi}_0^2 + \mathrm{O}\big(\delta\boldsymbol{\Psi}^2\big)\ . \end{aligned} \tag{4.51}$$

Considering (4.51) and by taking Lemma 13 into account, (4.29) becomes

$$\mathrm{tr}\,\boldsymbol{\Psi}_0\mathrm{tr}\,\delta\boldsymbol{\Psi} - \mathrm{tr}\,[\boldsymbol{\Psi}_0\delta\boldsymbol{\Psi}] = 0\,. \qquad \diamondsuit$$

Now, by applying the known relations for ψ_0 (Theorem 4), strictly speaking for $\boldsymbol{\Psi}_0$ (Lemma 13), to the partial differential equation presented in Lemma 14, the following form for the field equation in gravity space is derived:

Lemma 15 *Substitution of (4.42) into (4.50) leads to the linear field equation for the adjoint disturbing potential*

$$3\boldsymbol{\xi}^\top\delta\boldsymbol{\Psi}\boldsymbol{\xi} + \|\boldsymbol{\xi}\|^2\mathrm{tr}\,\delta\boldsymbol{\Psi} = 0\,. \tag{4.52}$$

Proof. To begin with, expressing (4.50) in index notation yields

$$\Psi^0_{ii}\delta\Psi_{kk} - \Psi^0_{ik}\delta\Psi_{ki} = 0\,. \tag{4.53}$$

Substitution of (4.42) in (4.53) leads to

$$\left[\delta_{ii} - \frac{3}{2}\frac{\xi_i\xi_i}{\|\boldsymbol{\xi}\|^2}\right]\delta\Psi_{kk} - \left[\delta_{ik} - \frac{3}{2}\frac{\xi_i\xi_k}{\|\boldsymbol{\xi}\|^2}\right]\delta\Psi_{ki} = 0\,, \tag{4.54}$$

and consequently to

$$6\|\boldsymbol{\xi}\|^2\delta\Psi_{kk} - 3\xi_i\xi_i\delta\Psi_{kk} - 2\|\boldsymbol{\xi}\|^2\delta\Psi_{ii} + 3\xi_i\delta\Psi_{ik}\xi_k = 0\,, \tag{4.55}$$

by multiplication with the factor $2\|\boldsymbol{\xi}\|^2$ and by writing out (4.54) explicitely. Now, the first three terms in (4.55) can be summarized as

$$\big(6\|\boldsymbol{\xi}\|^2 - 3\|\boldsymbol{\xi}\|^2 - 2\|\boldsymbol{\xi}\|^2\big)\,\delta\Psi_{kk} + 3\xi_i\delta\Psi_{ik}\xi_k = 0\,. \tag{4.56}$$

Finally, in reverse order and by using vector-matrix notation, (4.56) becomes

$$3\boldsymbol{\xi}^\top\delta\boldsymbol{\Psi}\boldsymbol{\xi} + \|\boldsymbol{\xi}\|^2\mathrm{tr}\,\delta\boldsymbol{\Psi} = 0\,. \qquad \diamondsuit$$

Remark 20 It can be emphasized that the adjoint disturbing potential $\delta\psi$ does not satisfy Laplace's equation in gravity space. In fact, this finding is in contrast to the result given in [66] Moritz 1980. Starting point in the argumentation of H. Moritz is that at corresponding boundary surfaces in gravity space and in ordinary space the adjoint disturbing potential $\delta\psi$ is simply the negative of the disturbing gravitational potential δV. This is admittedly true. However, a generalization with respect to the entire domain is not permissible. Therefore, the conclusion that if δV satisfies Laplace's equation in ordinary space, $\delta\psi$ has to do the same in gravity space, is not justifiable either. This reasoning can also be verified by the following counter example. Granted that the adjoint disturbing potential satisfies Laplace's equation $\Delta\delta\psi = 0$ in gravity space results due to (4.52) inevitably to

$$\boldsymbol{\xi}^\top\delta\boldsymbol{\Psi}\boldsymbol{\xi} = 0$$

by introducing $\Delta\delta\psi = \operatorname{tr}\delta\boldsymbol{\Psi} = 0$, cf. (2.31), in (4.52). Since $\boldsymbol{\xi}^\top\delta\boldsymbol{\Psi}\boldsymbol{\xi} = 0$ has to hold for any $\boldsymbol{\xi}$, at least in the interior domain Ω_g^-, cf. Fig. 4.1, there can only exist a solution if

$$\delta\boldsymbol{\Psi} = 0$$

holds, which corresponds to the trivial solution for $\delta\psi$

$$\delta\psi = 0\,.$$

Since the trivial solution is to be excluded from the solution domain, $\delta\psi$, as a matter of fact, does not satisfy Laplace's equation.

Hence, after having observed the correctness of the partial differential equation (4.52), which the adjoint disturbing potential $\delta\psi(\boldsymbol{\xi})$ has to fulfill, follows an investigation regarding the corresponding boundary condition. From (4.35), the boundary relation for the nonlinear problem is recalled

$$\left(\boldsymbol{\xi}^\top\nabla_\xi\psi - \psi\right)\Big|_\Sigma = v\,. \tag{4.57}$$

As before, Σ represents the image of the Earth's surface σ under the gravity space mapping (4.5). Formally, in view of Theorem 3, a similar relation results if instead of the true potential functions V and ψ, the normal potential functions V_0 and ψ_0 are considered

$$\left(\boldsymbol{\xi}^\top\nabla_\xi\psi_0 - \psi_0\right)\Big|_\Sigma = v_0\,. \tag{4.58}$$

Hence, in a first step it is possible, by numerical evaluation of (4.58), to compute the corresponding boundary values v_0 for the adjoint normal potential ψ_0. Now, the boundary values v_0 are to be interpreted as the restriction of the normal gravitational potential V_0 to a certain reference surface denoted by σ_0

$$v_0 = V_0\big|_{\sigma_0}\,.$$

The surface σ_0 can be understood as the image of Σ, if the normal gravitational potential V_0 instead of the true gravitational potential V, ψ_0 instead of ψ respectively, enters into the gravity space mapping process. However, the physical meaning of σ_0 and how to actually establish this surface in ordinary space remains temporarily unknown. Now, since the boundary conditions (4.57), (4.58) are both linear in ψ and ψ_0, subtraction of these two equations results in the boundary condition for the adjoint disturbing potential

$$\left(\boldsymbol{\xi}^\top\nabla_\xi\delta\psi - \delta\psi\right)\Big|_\Sigma = \Delta v\,. \tag{4.59}$$

Consequently, the corresponding boundary values, i.e. gravitational potential anomalies Δv, are obtained in a second step as the difference of the measured data v and the values v_0 deduced from evaluation of (4.58)

$$\Delta v = v - v_0 = v - V_0\big|_{\sigma_0}\,. \tag{4.60}$$

In conclusion, in order to derive the boundary data for the linear problem, a detour via gravity space is necessary. Furthermore, normal gravitational potential data are obtained for a surface σ_0, which was found not to be meaningful at first sight. Hence, for practical applications these circumstances might be deterring. However, remedy can be found by introducing the so-called *gravimetric telluroid*, which will shed light on the meaning of σ_0 and thus will provide a better understanding of the gravity space concept itself.

To take advantage of the fact that the gravity space coordinates $\boldsymbol{\xi}$ were found to be of gradient type (4.8), it is advantageous to abandon the realization of the telluroid Σ given previously in Definition 6 in terms of the potential type Marussi coordinates. Thus, the new concept for defining the telluroid surface is given by:

Definition 18 *The locus of points Q satisfying the condition*

$$\mathbf{g}_0(Q) = \mathbf{g}_0\big|_{\Sigma_g} = \mathbf{g}(P) = \mathbf{g}\big|_\sigma \tag{4.61}$$

is referred to as gravimetric telluroid and is denoted by Σ_g.

Hence, the normal gravitational acceleration vectors in points Q at Σ_g equal the actual gravitational acceleration vectors in points P situated at the Earth's topography σ. Furthermore, in the same way as σ is described by the position vector $\mathbf{x}\big|_\sigma$, the surface Σ_g is characterized by the position vector $\mathbf{x}\big|_{\Sigma_g}$. Note that Σ_g is generally based on a Somigliana-Pizzetti type of reference potential. However, in accordance with Theorem 4, the isotropic potential (2.40) is chosen as the convenient reference potential in the present case under. It will turn out that Σ_g, when defined in the described manner, resembles the reference surface σ_0 for the normal potential data v_0 in ordinary space.

Against the background of gravity space theory, where the components of the gravitational acceleration vector are introduced as new spatial coordinates in gravity space, (4.5), it is understandable that equal numerical values of $\mathbf{g}_0\big|_{\Sigma_g}$ and $\mathbf{g}\big|_\sigma$, cf. (4.61), correspond to the same point in gravity space. That is, for a topography point P specified by its position vector $\mathbf{x}\big|_\sigma$ and for its associated gravitational acceleration vector $\mathbf{g}(\mathbf{x}\big|_\sigma)$, the transformation from ordinary to gravity space, cf. (4.5), is explicitely given by

$$\boldsymbol{\xi}\big|_\Sigma = \mathbf{g}(\mathbf{x}\big|_\sigma)\,,$$

which is already stated in (4.6). Similarly, for a point Q of the gravimetric telluroid specified by its position vector $\mathbf{x}\big|_{\Sigma_g}$ and for its associated normal gravitational acceleration vector $\mathbf{g}_0(\mathbf{x}\big|_{\Sigma_g})$, the following transformation from ordinary to gravity space can be formulated

$$\boldsymbol{\xi}\big|_\Sigma = \mathbf{g}_0(\mathbf{x}\big|_{\Sigma_g})\,. \tag{4.62}$$

In other words, the Earth's surface σ and the gravimetric telluroid Σ_g, defined according to (4.61), are mapped onto the same surface Σ in gravity space by taking into consideration either the gradient of the true potential V in the points P on σ, or the gradient of the normal potential V_0 in the points Q on Σ_g.

Thus, taking (4.62) into account, it can be concluded that the gravimetric telluroid surface Σ_g according to (4.61) is consistent with the boundary condition (4.58). More precisely, in the context of (4.58) the surface σ_0 has been understood as the image of Σ if the normal instead of the true potential enters into the gravity space mapping process. However, it holds true that this interpretation corresponds to the relationship given by (4.62), only that the transformation is considered to be carried out in an inverse manner, i.e. from Σ to Σ_g. Consequently, the surfaces σ_0 and Σ_g involved in (4.58) and (4.62) must be one and the same surface. Moreover, σ_0, in the first instance found to be uninterpretable, can indeed be identified as the gravimetric telluroid Σ_g introduced in Definition 18. Furthermore, concerning the boundary data derivation, it can be claimed that the detour via gravity space becomes unnecessary, since after having established the gravimetric telluroid in agreement with (4.61), the normal gravitational potential values $V_0\big|_{\sigma_0} = V_0\big|_{\Sigma_g}$ can be computed directly in ordinary space.

Hence, by reordering the expression given in Lemma 15, i.e. the field equation for adjoint disturbing potential, thereby writing $\Delta\delta\psi$ instead of $\operatorname{tr}\delta\boldsymbol{\Psi}$, and by taking the the relations derived for the boundary condition (4.59) and the boundary data (4.60) into account, it can finally be concluded that the adjoint disturbing potential $\delta\psi(\boldsymbol{\xi})$ solves a BVP defined as follows:

Definition 19 *The linear geodetic boundary value problem in gravity space represents the following problem: the data, i.e. gravitational potential anomalies Δv, are considered to be given at the known surface Σ*

$$\Delta v : \Sigma \to \mathbb{R}$$

and required is a real function $\delta\psi(\boldsymbol{\xi}) : int\ \Sigma \to \mathbb{R}$

$$\Delta\delta\psi = \frac{-3}{\|\boldsymbol{\xi}\|^2}\boldsymbol{\xi}^\top\delta\boldsymbol{\Psi}\boldsymbol{\xi} \tag{4.63}$$

$$\left(\boldsymbol{\xi}^\top\nabla_\xi\delta\psi - \delta\psi\right)\Big|_\Sigma = \Delta v\,, \tag{4.64}$$

which is the solution of a linear but inhomogeneous partial differential equation of second order, (4.63), under the boundary condition (4.64).

With respect to Definition 19, it must be added that (4.63) and (4.64) establish a *fixed* BVP in gravity space based on a inharmonic differential equation and a linear oblique boundary condition. The known boundary surface Σ in gravity space is determined according to Definition 15.

In the first instance, by means of $\delta\psi(\boldsymbol{\xi})$, which results as the solution of the BVP characterized in Definition 19, the adjoint potential $\psi(\boldsymbol{\xi})$ can be derived according to (4.47). Furthermore, the unknown terrestrial topography σ can be determined. For that purpose, starting from (4.36) and taking (4.47) into account, the relation

$$\begin{aligned}\mathbf{x}\big|_\sigma = \nabla_\xi\psi(\boldsymbol{\xi}\big|_\Sigma) &= \nabla_\xi\left(\psi_0(\boldsymbol{\xi}\big|_\Sigma) + \delta\psi(\boldsymbol{\xi}\big|_\Sigma)\right)\\ &= \nabla_\xi\psi_0(\boldsymbol{\xi}\big|_\Sigma) + \nabla_\xi\delta\psi(\boldsymbol{\xi}\big|_\Sigma)\\ &= \mathbf{x}\big|_{\Sigma_g} + \boldsymbol{\zeta}\end{aligned} \tag{4.65}$$

is derived. In (4.65), use of the following identity

$$\mathbf{x}\big|_{\Sigma_g} = \nabla_\xi\psi_0(\boldsymbol{\xi}\big|_\Sigma)\,, \tag{4.66}$$

which is of similar mathematical structure as (4.36), has been made. Thus, (4.66) represents a gradient type relation, cf. also Lemma 9, and results, in a manner of speaking, from the associated gradient type expression (4.62). That is $\boldsymbol{\xi}\big|_\Sigma = \nabla V_0(\mathbf{x}\big|_{\Sigma_g})$, by exchanging ordinary and gravity space quantities. More precisely, taking advantage of the underlying properties of the Legendre transformation, Section 2-5.3, which as is known from Remark 16 forms the basis of F. Sansò's gravity space approach, yields (4.66) from (4.62) by exploiting the immanent mutual symmetry of position and momentum quantities. Hence, it can be concluded that from the adjoint disturbing potential $\delta\psi$ the position correction or position anomaly vector, see Fig. 3.1,

$$\boldsymbol{\zeta} = \nabla_\xi\delta\psi(\boldsymbol{\xi}\big|_\Sigma) \tag{4.67}$$

is obtained. Finally, according to (4.65), summation of the vector $\boldsymbol{\zeta}$ and the vector $\mathbf{x}\big|_{\Sigma_g}$ constituting the known approximating surface, i.e. the gravimetric telluroid Σ_g, the requested vector $\mathbf{x}\big|_\sigma$ is found, describing the Earth's topography σ.

Remark 21 Theoretically, the boundary data Δv or more precisely the difference on the right hand side of (4.60) can be evaluated a little further

$$\Delta v = V\big|_\sigma - V_0\big|_{\Sigma_g} \cong W\big|_\sigma - W_0\big|_{\Sigma_g} = \Delta w\,. \tag{4.68}$$

Here, the assumption has been made that the difference of the centrifugal potentials at the Earth's surface σ and the gravimetric telluroid Σ_g is negligibly small. As far as the linear problem is concerned, one consequence is that the correction of the rotational influence, see Definition 13, which is necessary as discussed in Section 4-1, can be neglected. Thus, comparable to the classical theory as presented in Chapter 3, it is possible to use the boundary data Δw as accessible by observations without applying further reductions. However, for reasons of accurateness and to avoid assumptions of any kind, the rotational free condition is retained unchanged in the following investigations.

4-6 Considerations on the asymptotic behaviour of the related functions

Under the assumption of large values for the spatial radius vector $\|\mathbf{x}\|$ the asymptotic relation for the gravitational potential V can be expressed by

$$V = \frac{GM}{\|\mathbf{x}\|} + \mathrm{O}\big(\|\mathbf{x}\|^{-2}\big) \tag{4.69}$$

and, consequently, the asymptotic behaviour of the modulus of the gravitational acceleration vector $\|\mathbf{g}\| = g$ is

$$g = \frac{GM}{\|\mathbf{x}\|^2} + \mathrm{O}\big(\|\mathbf{x}\|^{-3}\big)\,. \tag{4.70}$$

As already stated in Section 4-1 in the beginning of this chapter, for $\|\mathbf{x}\| \to \infty$ in (4.70) results $g \to 0$, so that spatial infinity corresponds to the origin in gravity space. Next, similarly to (4.40), using (4.69) in (4.19) yields

$$\psi = -2\frac{GM}{\|\mathbf{x}\|} + \mathrm{O}\big(\|\mathbf{x}\|^{-2}\big)\,. \tag{4.71}$$

Now, solving (4.70) for $\|\mathbf{x}\|$, thereby taking $\|\boldsymbol{\xi}\| = g$, (4.5), into account

$$\|\mathbf{x}\| = \frac{\sqrt{GM}}{\sqrt{g}} + \mathrm{O}\Big(g^{-3/2}\Big) = \frac{\sqrt{GM}}{\sqrt{\|\boldsymbol{\xi}\|}} + \mathrm{O}\Big(\|\boldsymbol{\xi}\|^{-3/2}\Big) \tag{4.72}$$

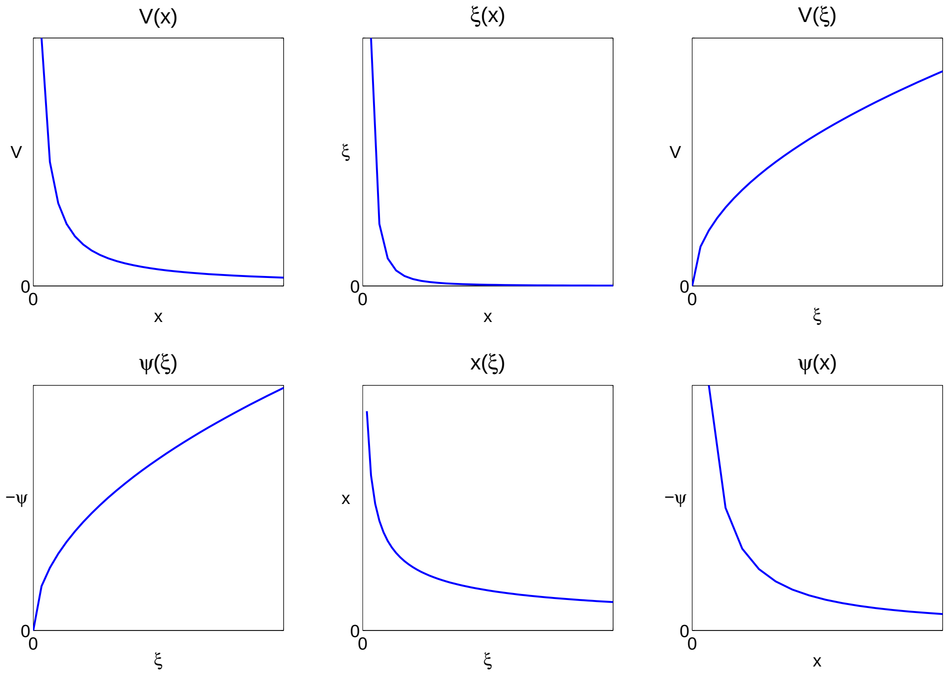


Figure 4.3: Asymptotic relations (linear scaling).

and substituting (4.72) into (4.71) yields, cf. Theorem 4,

$$\psi = -2\sqrt{GM\|\boldsymbol{\xi}\|} + \mathrm{O}\left(\|\boldsymbol{\xi}\|^{3/2}\right) . \tag{4.73}$$

Analogously to (4.73), insertion of (4.72) into (4.69) results in

$$V = \sqrt{GM\|\boldsymbol{\xi}\|} + \mathrm{O}\left(\|\boldsymbol{\xi}\|^{3/2}\right) . \tag{4.74}$$

Finally, it should be pointed out that for $\|\boldsymbol{\xi}\| \to 0$ in (4.73) results directly $\psi \to 0$. This means that at the origin of gravity space the adjoint potential vanishes. This also becomes obvious in Fig. 4.3, where the above asymptotic relations are plotted using a linear scale.

The question to be settled next is what are the consequences of $\psi(\mathbf{0}) = 0$ for the solution of the BVP (4.34)-(4.35). Following (4.34), to come up with an answer, (4.73) has to be differentiated twice in order to obtain the corresponding matrix $\boldsymbol{\Psi}$, cf. Lemma 10 and Lemma 13,

$$\boldsymbol{\Psi} = [\Psi_{ij}] = \frac{-\sqrt{GM}}{\|\boldsymbol{\xi}\|^{3/2}} \left[\delta_{ij} - \frac{3}{2}\frac{\xi_i \xi_j}{\|\boldsymbol{\xi}\|^2}\right] + \mathrm{O}\left(\|\boldsymbol{\xi}\|^{-1/2}\right) . \tag{4.75}$$

Hence, due to the fraction $\|\boldsymbol{\xi}\|^{-3/2}$ in (4.75), it can be concluded that for $\|\boldsymbol{\xi}\| \to 0$ every solution of the BVP (4.34)-(4.35) possesses a singularity at the origin, see [79] Sansò 1977. Since the concept of differentiability has no meaning at spatial infinity in ordinary space, it is not a surprise that in the new gravity space coordinates $\boldsymbol{\xi}$, differentiability is lost at the origin. This singularity considerably reduces the mathematical beauty of F. Sansò's gravity space approach. Hence, the motivation is to directly proceed with an approach, which will overcome this restriction.

Remark 22 For the sake of completeness, it has to be admitted that F. Sansò proposed a possibility to eliminate this singularity problem. That is to say, in [79] Sansò 1977, he suggests, by means of specific weighting, a further change of variables involving new gravity space coordinates, which do not exhibit any singularities, and a new so-called *reduced* adjoint potential, see [81] Sansò 1978. Of course, within the linearization process the reduced adjoint potential results in a new adjoint disturbing potential. Fortunately, this new adjoint disturbing potential is now in fact harmonic in the region of interest. Hence, the field equation of the corresponding BVP becomes Laplace's equation and the overall BVP resembles a Dirichlet problem, see [85] Sansò 1981. However, this tedious and extra labor in terms of multiple substitutions is not actually necessary, as will be shown in the next chapter.

Chapter 5

A boundary value approach in regular gravity space

Motivated by the necessity to overcome the addressed shortcomings inherent in F. Sansò's gravity space approach as outlined in the previous chapter, a revised gravity space approach will be introduced in this chapter. The new approach relies on a so-called *singularity-free* or *regular* gravity space transformation and is essentially based on the contributions of W. Keller, e.g. [46],[47],[48] Keller 1986,1987. Organization of the presented material is chosen in the preferred manner following the example of the last two chapters. At first, the new gravity space approach is established and its associated properties are elaborated, see Section 5-1. Next, the resulting nonlinear fixed BVP in gravity space is deduced in Section 5-2. Thereafter, in Section 5-3, linearization of the BVP is conducted. Section 5-4 is devoted to the task of developing a representation of the considered problem in spherical approximation. Further modification of the underlying BVP in terms of a constant radius approximation is accomplished in Section 5-5, which leads to the final form of the GBVP in regular gravity space required in view of the numerical solution. A discussion on the theoretical and practical advantages and drawbacks of the presented gravity space theories and a comparison with the classical Molodensky theory concludes this chapter.

5-1 A singularity-free gravity space transformation

In the previous chapter F. Sansò's gravity space approach has been found to hold a singularity at the origin. The reason for this shortcoming has been identified in the mapping of a point at spatial infinity to the origin of gravity space, which led to a loss of differentiability. Thus, this singularity can only be avoided by a similar gravity space transformation, which leaves the point at infinity as a fixed point and therefore guarantees differentiability. Such a gravity space transformation, referred to as the regular gravity space transformation from now on, has been elaborated first by W. Keller in e.g. [48] Keller 1987.
As a start, the approach proposed is presented involving the new coordinate transformation and the forward relation to obtain the adjoint potential from the gravitational potential. The asymptotic behaviour of the underlying regular gravity space transformation is observed in the first paragraph thereafter, see Section 5-1.1. A next paragraph, Section 5-1.2, comprises a lemma devoted to the issue of *identical mapping*, and another lemma related to a revised definition of the *gravimetric telluroid*, thereby highlighting important aspects of the approach. Furthermore, a theorem constitutes the backward transformation related to the potential functionals, that is to say the computation of the gravitational potential from the adjoint potential. In conjunction with this theorem, two lemmata provide the necessary Jacobian matrices. A last paragraph, which reconsiders the proposed gravity space approach against the background of contact transformations, provides the basis for the formulation of the nonlinear fixed exterior GBVP in gravity space in the section thereafter.
As mentioned above, the proposed revised gravity space approach is based on the new *regular gravity space* concept:

Definition 20 *Regular gravity space is the image of the Earth's exterior domain under the modified mapping*

$$\boldsymbol{\xi} = [\xi_i] := -\sqrt{GM}\frac{\nabla V(\mathbf{x})}{\|\nabla V(\mathbf{x})\|^{3/2}} \,. \tag{5.1}$$

The components ξ_i of the vector $\boldsymbol{\xi}$ constitute new independent Cartesian coordinates in regular gravity space.

Analogously to (4.8), (5.1) forms the new gravity space transformation, which maps a position vector $\mathbf{x}$ of the ordinary space to a vector $\boldsymbol{\xi}$ of the gravity space. A scrutinizing glance reveals that (5.1) is closely related to the former gravity space transformation (4.8). In fact, (5.1) roughly results from (4.8) by means of a special normalization designed in such a way that a point at infinity is left unchanged. In order to underline this transformation behaviour, the corresponding asymptotic relations, cf. Section 4-6, are addressed after the end of this discussion.

Next, a transition into regular gravity space, as was previously the case in F. Sansò's approach, involves a change of the underlying potential function. By definition, the corresponding adjoint potential is required to result from the following expression

$$\psi(\boldsymbol{\xi}) := \mathbf{x}^\top \nabla V(\mathbf{x}) - V(\mathbf{x}) \quad ; \quad \mathbf{x} = \mathbf{x}(\boldsymbol{\xi})\,. \tag{5.2}$$

Hence, (5.2), which relates the adjoint potential $\psi(\boldsymbol{\xi})$ to the gravitational potential $V(\mathbf{x})$, formally remains the same as in Definition 16. However, due to the fact that the adjoint potential is a function of $\boldsymbol{\xi}$, the coordinate vector in ordinary space $\mathbf{x}$ has to be interpreted as a function of $\boldsymbol{\xi}$ as well, i.e. $\mathbf{x}(\boldsymbol{\xi})$. In addition, since different coordinate vectors $\boldsymbol{\xi}$ in gravity space are deduced from (4.5) and (5.1), it is essential to bear in mind that the related adjoint potentials are indeed different. It should already be pointed out that similarly to F. Sansò's approach, the newly proposed regular gravity space approach also constitutes a contact transformation, as will be addressed in detail in a later paragraph. As mentioned before, the resulting asymptotic relations and the immanent properties of the new approach are first accounted for.

5-1.1 Considerations on the asymptotic behaviour of the related functions

As indicated above, in order to confirm the singularity-free character of the above transformations, (5.1) and (5.2), it is worthwhile to take a look at the asymptotic behaviour of the related functionals in ordinary and gravity space. At first, the following relationships

$$V = \frac{GM}{\|\mathbf{x}\|} + \mathrm{O}\big(\|\mathbf{x}\|^{-2}\big) \tag{5.3}$$

$$\|\nabla V\| = \frac{GM}{\|\mathbf{x}\|^2} + \mathrm{O}\big(\|\mathbf{x}\|^{-3}\big) \tag{5.4}$$

$$\psi = -2\frac{GM}{\|\mathbf{x}\|} + \mathrm{O}\big(\|\mathbf{x}\|^{-2}\big)\ . \tag{5.5}$$

have remained unchanged as given in Section 4-6. Now, by writing

$$\boldsymbol{\xi} = -\frac{\sqrt{GM}}{\sqrt{\|\nabla V(\mathbf{x})\|}}\frac{\nabla V(\mathbf{x})}{\|\nabla V(\mathbf{x})\|}\,,$$

for (5.1), the modulus of $\boldsymbol{\xi}$ follows directly as

$$\|\boldsymbol{\xi}\| = \frac{\sqrt{GM}}{\sqrt{\|\nabla V(\mathbf{x})\|}}\,. \tag{5.6}$$

Substitution of (5.4) into (5.6) yields

$$\|\boldsymbol{\xi}\| = \|\mathbf{x}\| + \mathrm{O}\Big(\|\mathbf{x}\|^{3/2}\Big)\ . \tag{5.7}$$

This finding immediately leads to the remaining two asymptotic relations

$$V = \frac{GM}{\|\boldsymbol{\xi}\|} + \mathrm{O}\Big(\|\boldsymbol{\xi}\|^{-3/2}\Big) \tag{5.8}$$

and

$$\psi = -2\frac{GM}{\|\boldsymbol{\xi}\|} + \mathrm{O}\Big(\|\boldsymbol{\xi}\|^{-3/2}\Big)\ . \tag{5.9}$$

Now, for $\|\mathbf{x}\| \to \infty$, $\|\boldsymbol{\xi}\| \to \infty$ can be deduced directly from the important relationship (5.7), which states that the norm of $\boldsymbol{\xi}$ equals the norm of $\mathbf{x}$ in a first approximation. This means that by the use of (5.1) a point at spatial infinity in ordinary space also remains fixed at infinity in gravity space. This contrasts with the previous gravity space transformation according to F. Sansò as has been worked out in Section 4-6. Furthermore, taking $\|\boldsymbol{\xi}\| \to \infty$ in (5.9), results in $\psi \to 0$. Thus, the asymptotic behaviour of the adjoint potential in gravity space now nearly

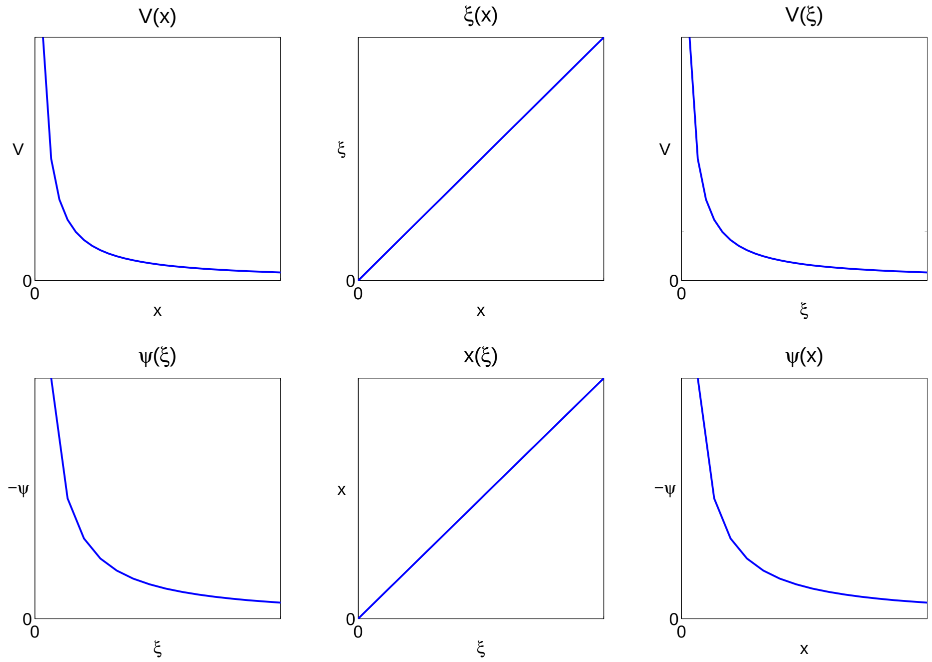


Figure 5.1: Asymptotic relations (linear scaling).

resembles the asymptotic behaviour of the gravitational potential in ordinary space. Moreover, it can be concluded that the resulting BVP in gravity space constitutes an exterior problem and the aspect of losing differentiability at the origin is therefore no longer an issue. Consequently, the new gravity space transformations (5.1), (5.2) meet the requirements given at the beginning of this section in order to define a *regular* approach. Finally, the asymptotic relations provided above are shown in Fig. 5.1 using a linear scale.

5-1.2 Important properties of the regular gravity space approach

After having confirmed in the preceding paragraph that the new gravity space approach according to (5.1) and (5.2) does not exhibit any singularity problem, this paragraph deals with two relevant properties of the newly proposed method. That is, the special case of identical mapping, which provides a deeper insight into the new transformation and the concept of gravimetric telluroid mapping in gravity space. The latter is applied to obtain the boundary surface in gravity space from gravity observations at the Earth's surface. Furthermore, the equation to derive the gravitational potential from the adjoint potential will be derived. As seen before, this relationship will be essential for the setup of the boundary condition.
Prompted by the recovery of a strong similarity in the asymptotic relations of ordinary and gravity space functionals, cf. Fig. 5.1, the following instructive problem is treated in the context of the next lemma. That is, consideration of the scenario where the true potential V is reduced to the isotropic normal potential V_0, i.e. to the gravitational potential of a point mass or of a sphere with radial mass density distribution:

Lemma 16 *Let*

$$V(\mathbf{x}) = V_0(\mathbf{x}) \qquad \textit{with} \qquad V_0(\mathbf{x}) = \frac{GM}{\|\mathbf{x}\|}\,, \tag{5.10}$$

then the relationship of independent coordinates in geometry space and regular gravity space simply becomes

$$\boldsymbol{\xi} = \mathbf{x}\,. \tag{5.11}$$

Proof. The assertion of the above lemma follows directly from substituting (5.10) in (5.1)

$$\boldsymbol{\xi} = -\sqrt{GM}\,\frac{\nabla V_0(\mathbf{x})}{\|\nabla V_0(\mathbf{x})\|^{3/2}} = -\sqrt{GM}\,\frac{-\frac{GM}{\|\mathbf{x}\|^3}\mathbf{x}}{\|\frac{GM}{\|\mathbf{x}\|^2}\frac{\mathbf{x}}{\|\mathbf{x}\|}\|^{3/2}} = \frac{\frac{GM^{3/2}}{\|\mathbf{x}\|^3}\mathbf{x}}{\frac{GM^{3/2}}{\|\mathbf{x}\|^3}} = \mathbf{x}\,. \qquad \diamondsuit$$

The special case of Lemma 16 is referred to as *identical mapping* and the identity (5.11) is of course in accordance with (5.7). Moreover, (5.11) suggests a gravity space mapping scenario as illustrated in Fig. 5.2.

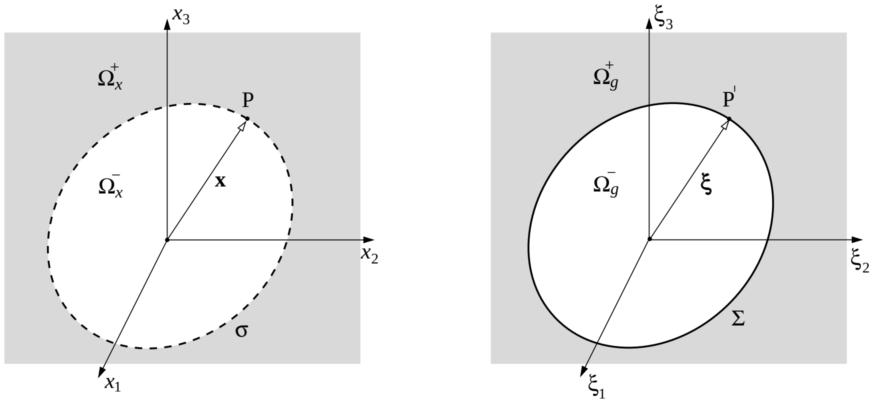


Figure 5.2: Regular gravity space mapping.

In Fig. 5.2, both images, one representing the configuration in ordinary space, the other the configuration in gravity space, must be regarded as congruent. This is a direct result of the identical mapping $\boldsymbol{\xi} = \mathbf{x}$ according to Lemma 16, and means that in terms of dimensions and units, the situation in gravity space now exactly matches the situation in ordinary space. Strictly speaking, the boundary surface σ representing the physical surface of the Earth becomes the boundary surface Σ in a one-to-one manner. However, in contrast to σ given in ordinary space, the boundary surface Σ in gravity space is known. Thus, the GBVP, which has been identified as a free exterior BVP in ordinary space, transforms into a fixed exterior BVP within the scope of a regular gravity space ansatz. Note the difference that the resulting fixed BVP represented an interior problem in the context of the gravity space approach according to F. Sansò presented in the last chapter.
Next, abandonment of the restriction that V is reduced to V_0, can be understood as a perturbation of the setup given in Fig. 5.2. However, the overall situation remains nearly the same, yet a difference results in the boundary surfaces σ and Σ. This circumstance leads to the next lemma, which addresses the problem of finding Σ in the general case:

Lemma 17 *With the isotropic normal potential given by*

$$V_0(\mathbf{x}) = \frac{GM}{\|\mathbf{x}\|}\,, \tag{5.12}$$

it holds

$$\nabla V_0(\boldsymbol{\xi}) = \nabla V(\mathbf{x})\,, \tag{5.13}$$

when V_0 is understood as a function symbol and not as a physical quantity.

Proof. Taking (5.1) into account yields

$$\nabla V_0(\boldsymbol{\xi}) = -GM \frac{-\sqrt{GM}\frac{\nabla V(\mathbf{x})}{\|\nabla V(\mathbf{x})\|^{3/2}}}{\|-\sqrt{GM}\frac{\nabla V(\mathbf{x})}{\|\nabla V(\mathbf{x})\|^{3/2}}\|^3} = \frac{\frac{GM^{3/2}}{\|\nabla V(\mathbf{x})\|^{3/2}}\nabla V(\mathbf{x})}{\left(\frac{\sqrt{GM}}{\sqrt{\|\nabla V(\mathbf{x})\|}}\right)^3} = \nabla V(\mathbf{x})\,. \qquad \diamondsuit$$

The remarkable property revealed within the scope of this lemma provides the basis for the next definition:

Definition 21 *By restricting the position vector $\mathbf{x}$ of geometry space onto the surface of the Earth, i.e. $\mathbf{x}\big|_\sigma$, and by the use of Lemma 17, the corresponding position vector $\boldsymbol{\xi}\big|_\Sigma$ for the gravity space boundary surface Σ is obtained if the following condition holds*

$$\nabla V_0(\boldsymbol{\xi}\big|_\Sigma) = \nabla V(\mathbf{x}\big|_\sigma)\,. \tag{5.14}$$

Since (5.14) is formally identical to (4.61) given in Definition 18, the boundary surface in gravity space Σ is also referred to as gravimetric telluroid.

Thus, it is important to distinguish that the gravimetric telluroid Σ_g, as introduced by (4.61) at the end of Section 4-5, has been defined as a surface in geometry space, whereas the gravimetric telluroid Σ according to last definition establishes a reference surface in gravity space. Nevertheless, due to the affinity of (4.61) and (5.14), the surfaces Σ_g and Σ are geometrically identical. Furthermore, the definition of the surface Σ according to Definition 21 is of course equivalent to the universal definition that the surface Σ is the image of the Earth's surface σ under the mapping (5.1).

In order to formulate the boundary condition of the nonlinear GBVP in regular gravity space as will be done in the next section, knowledge is mandatory, not only about the boundary surface, but also about the corresponding boundary data. Now, since the last lemma has already addressed the problem of finding the boundary surface in regular gravity space, finding the corresponding boundary values is the question to be settled next. The inverse transformation of (5.2) is therefore derived in the context of the next theorem. To perform this task, two preliminary lemmata, which are providing the necessary Jacobian matrices of the coordinate transformations $\boldsymbol{\xi} = \boldsymbol{\xi}(\mathbf{x})$ and $\mathbf{x} = \mathbf{x}(\boldsymbol{\xi})$, are given first:

Lemma 18 *For the Jacobian matrix of the coordinate transformation $\boldsymbol{\xi} = \boldsymbol{\xi}(\mathbf{x})$ holds*

$$\left[\frac{\partial \xi_i}{\partial x_j}\right] = -\frac{\sqrt{GM}}{\|\nabla V\|^{3/2}}\left[\nabla(\nabla V)\right]\left[\mathbf{I} - \frac{3}{2}\frac{\nabla V \nabla V^\top}{\|\nabla V\|^2}\right]. \tag{5.15}$$

Proof. The statement of Lemma 18 is received straightforwardly from (5.1) by means of evaluation according to the chain rule of differential calculus

$$\begin{aligned}\left[\frac{\partial \xi_i}{\partial x_j}\right] &= -\sqrt{GM}\frac{\left[\nabla(\nabla V)\right]\|\nabla V\|^{3/2} - \frac{3}{2}\nabla V\|\nabla V\|^{-1/2}(\nabla V)^\top\left[\nabla(\nabla V)\right]}{\|\nabla V\|^3} \\ &= -\frac{\sqrt{GM}}{\|\nabla V\|^{3/2}}\left[\nabla(\nabla V)\right]\left[\mathbf{I} - \frac{3}{2}\frac{\nabla V \nabla V^\top}{\|\nabla V\|^2}\right]. \qquad \diamondsuit\end{aligned}$$

Lemma 19 *Accordingly, the Jacobian matrix of the inverse coordinate transformation $\mathbf{x} = \mathbf{x}(\boldsymbol{\xi})$ is given by*

$$\left[\frac{\partial x_j}{\partial \xi_i}\right] = -\frac{\|\nabla V\|^{3/2}}{\sqrt{GM}}\left[\mathbf{I} - 3\frac{\nabla V \nabla V^\top}{\|\nabla V\|^2}\right]\left[\nabla(\nabla V)\right]^{-1} = \left[\frac{\partial \xi_i}{\partial x_j}\right]^{-1}. \tag{5.16}$$

Proof. In order to demonstrate that the matrix $\left[\frac{\partial x_i}{\partial \xi_j}\right]$ of the transformation $\mathbf{x} = \mathbf{x}(\boldsymbol{\xi})$ is indeed the Jacobian matrix of the inverse transformation $\boldsymbol{\xi} = \boldsymbol{\xi}(\mathbf{x})$, it is sufficient to show that according to Lemma 1

$$\left[\frac{\partial \xi_i}{\partial x_j}\right]\left[\frac{\partial x_j}{\partial \xi_k}\right] = [\delta_{ik}] = \mathbf{I} \tag{5.17}$$

holds. Consequently, in view of (5.15) and (5.16), the associated matrix multiplication yields

$$\begin{aligned}&\frac{-\sqrt{GM}}{\|\nabla V\|^{3/2}}\frac{-\|\nabla V\|^{3/2}}{\sqrt{GM}}\left[\nabla(\nabla V)\right]\left[\mathbf{I} - \frac{3}{2}\frac{\nabla V \nabla V^\top}{\|\nabla V\|^2}\right]\left[\mathbf{I} - 3\frac{\nabla V \nabla V^\top}{\|\nabla V\|^2}\right]\left[\nabla(\nabla V)\right]^{-1} = \\ &= \left[\left[\nabla(\nabla \mathbf{V})\right] - \frac{3}{2}\left[\nabla(\nabla V)\right]\frac{\nabla V \nabla V^\top}{\|\nabla V\|^2}\right]\left[\left[\nabla(\nabla \mathbf{V})\right]^{-\mathbf{1}} - 3\left[\nabla(\nabla V)\right]^{-1}\frac{\nabla V \nabla V^\top}{\|\nabla V\|^2}\right] \\ &= \mathbf{I} - 3\frac{\nabla V \nabla V^\top}{\|\nabla V\|^2} - \frac{3}{2}\frac{\nabla V \nabla V^\top}{\|\nabla V\|^2} + \frac{9}{2}\frac{\nabla V \nabla V^\top \nabla V \nabla V^\top}{\|\nabla V\|^4} \\ &= \mathbf{I} - \frac{9}{2}\frac{\nabla V \nabla V^\top}{\|\nabla V\|^2} + \frac{9}{2}\frac{\nabla V \nabla V^\top}{\|\nabla V\|^2} = \mathbf{I}. \qquad \diamondsuit\end{aligned}$$

It is worth noting that the Jacobian matrices introduced here are obviously closely related to the matrices $\mathbf{V}$ and $\boldsymbol{\Psi}$ given by (4.16) in Section 4-3 and by (4.27) in Section 4-4. Moreover, they provide the basis for the theorem below. This theorem constitutes the aforementioned backward transformation to compute the gravitational potential from the adjoint potential, which is necessary to establish the required boundary condition later on:

Theorem 5 *The inverse transformation of (5.2) reads as*

$$V(\mathbf{x}) = -\left(\frac{1}{2}\boldsymbol{\xi}^\top \nabla_\xi \psi(\boldsymbol{\xi}) + \psi(\boldsymbol{\xi})\right) \quad ; \quad \boldsymbol{\xi} = \boldsymbol{\xi}(\mathbf{x}). \tag{5.18}$$

Proof. Starting from (5.2), the gradient of the adjoint potential is obtained by application of the chain rule

$$\nabla_\xi \psi(\boldsymbol{\xi}) = \nabla \left(\mathbf{x}^\top \nabla V(\mathbf{x}) - V(\mathbf{x})\right) \left[\frac{\partial \mathbf{x}}{\partial \boldsymbol{\xi}}\right]$$

and consequently

$$\begin{aligned} \nabla_\xi \psi &= \left(\mathbf{x}^\top \left[\nabla(\nabla V)\right] + \nabla V - \nabla V\right) \left[\frac{\partial \boldsymbol{\xi}}{\partial \mathbf{x}}\right]^{-1} \\ &= \left(\mathbf{x}^\top \left[\nabla(\nabla V)\right]\right) \left[\frac{\partial \boldsymbol{\xi}}{\partial \mathbf{x}}\right]^{-1}, \end{aligned} \tag{5.19}$$

by omitting the identifiers $(\boldsymbol{\xi}), (\mathbf{x})$ and using Lemma 19. Now, substitution of $\boldsymbol{\xi}$ and ψ in

$$-\left(\frac{1}{2}\boldsymbol{\xi}^\top \nabla_\xi \psi + \psi\right)$$

according to (5.1) and (5.2), again neglecting the identifier $(\mathbf{x})$, results in

$$-\left(\frac{1}{2}\boldsymbol{\xi}^\top \nabla_\xi \psi + \psi\right) = \frac{1}{2}\sqrt{GM}\frac{\nabla V^\top}{\|\nabla V\|^{3/2}} \nabla_\xi \psi - \mathbf{x}^\top \nabla V + V\,. \tag{5.20}$$

Next, with (5.19) in (5.20)

$$-\left(\frac{1}{2}\boldsymbol{\xi}^\top \nabla_\xi \psi + \psi\right) = \frac{1}{2}\sqrt{GM}\frac{\nabla V^\top}{\|\nabla V\|^{3/2}} \left(\mathbf{x}^\top \left[\nabla(\nabla V)\right]\right) \left[\frac{\partial \boldsymbol{\xi}}{\partial \mathbf{x}}\right]^{-1} - \mathbf{x}^\top \nabla V + V \tag{5.21}$$

is deduced. At last, insertion of the Jacobian matrix according to Lemma 19 into (5.21) yields

$$-\left(\frac{1}{2}\boldsymbol{\xi}^\top \nabla_\xi \psi + \psi\right) =$$

$$= -\frac{1}{2}\sqrt{GM}\frac{\nabla V^\top}{\|\nabla V\|^{3/2}} \left(\mathbf{x}^\top \left[\nabla(\nabla V)\right]\right) \left(\frac{\|\nabla V\|^{3/2}}{\sqrt{GM}}\left[\mathbf{I} - 3\frac{\nabla V \nabla V^\top}{\|\nabla V\|^2}\right]\left[\nabla(\nabla V)\right]^{-1}\right) - \mathbf{x}^\top \nabla V + V\,. \tag{5.22}$$

Taking advantage of the symmetry of the matrices involved and simplifying (5.22) leads to

$$-\left(\frac{1}{2}\boldsymbol{\xi}^\top \nabla_\xi \psi + \psi\right) = -\frac{1}{2}\mathbf{x}^\top \left[\mathbf{I} - 3\frac{\nabla V \nabla V^\top}{\|\nabla V\|^2}\right] \nabla V - \mathbf{x}^\top \nabla V + V \tag{5.23}$$

and finally to

$$-\left(\frac{1}{2}\boldsymbol{\xi}^\top \nabla_\xi \psi + \psi\right) = -\frac{1}{2}\mathbf{x}^\top \nabla V + \frac{3}{2}\mathbf{x}^\top \nabla V \frac{\nabla V^\top \nabla V}{\|\nabla V\|^2} - \mathbf{x}^\top \nabla V + V = \mathbf{x}^\top \nabla V(-\frac{1}{2} + \frac{3}{2} - 1) + V = V\,. \qquad \diamondsuit$$

It is worth mentioning that (5.18) can also be verified by the reasoning given later within the scope of (5.66) and (5.67). Nevertheless, the mutual symmetry of the equations (5.2) and (5.18), relating adjoint potential ψ and ordinary potential V to one another, is obviously lost due to the presence of the factor $\frac{1}{2}$ and the double minus sign, which is in contrast to the previous result in terms of Definition 16 and Theorem 3 in case of F. Sansò's gravity space approach. Consequently, it can be concluded that the underlying transformations for the potential relations (5.2) and (5.18), in case of the regular gravity space approach, are not in agreement with the defining equations (2.88) and (2.92) of a Legendre transformation. However, even if the transformations involved in the context of the regular gravity space approach do not constitute a Legendre transformation, they still fulfill the requirements in order to establish a contact transformation, as will be presented in the next paragraph.

5-1.3 Contact transformation representation

The aim of this paragraph is to give evidence that W. Keller's regular gravity space approach is based on a contact transformation as specified in (2.82) and (2.83) of Definition 2. Hence, a close relationship to Section 2-5 and Remark 16 can be pointed out, the latter essentially identifying F. Sansò's gravity space approach as a Legendre transformation and thus as a member of the class of contact transformations. For the intended purpose, both gravity space approaches discussed so far are shortly reviewed and rewritten in a slightly different representation. Three lemmata, which are concerned with the functional matrices of the transformations between coordinates and momentum variables, are presented. They are preparatory, together with two more remarks, for a final lemma dealing with the Jacobian related to the new transformation approach and setting the stage for a final theorem eventually identifying W. Keller's new set of transformation formulae as a member of the family of contact transformations.

In order to get started, it is useful to contrast the approaches of F. Sansò and W. Keller, thereby unifying the notation. That is, as opposed to (2.82), the two coordinate vectors before and after transformation are henceforth denoted by $\mathbf{x}$ and $\boldsymbol{\xi}$ instead of $\mathbf{X}$. Furthermore, the two vectors of the old and new momentum variables are referred to as $\mathbf{p}$ and $\boldsymbol{\pi}$ instead of $\mathbf{P}$. At last, the corresponding potential functions are identified as V and ψ, instead of u and U. Then, from Section 2-5 and from Section 4-3 it is recapitulated that in the course of F. Sansò's approach, the potential V, the coordinates $\mathbf{x}$ and the momentum variables $\mathbf{p} = \nabla V(\mathbf{x})$ are transformed by means of a Legendre transformation into new quantities, i.e. into the adjoint potential ψ, the new coordinates $\boldsymbol{\xi}$ and new momenta $\boldsymbol{\pi} = \nabla_\xi \psi(\boldsymbol{\xi})$, in the familiar manner described by

$$
\begin{aligned}
\boldsymbol{\xi} &= \mathbf{p} && (5.24)\\
\psi &= \mathbf{p}^\top \mathbf{x} - V(\mathbf{x}) && (5.25)\\
\boldsymbol{\pi} &= \mathbf{x}\,. && (5.26)
\end{aligned}
$$

Furthermore, it is recalled that as validated in Section 2-5.3, the Legendre transformation (5.24)-(5.26) belongs to the family of contact transformations. Once more, the speciality associated with such a transformation becomes clear by exploring relation (5.24) and (5.26). Thus it appears that the new coordinates $\boldsymbol{\xi}$ result from the old momentum variables $\mathbf{p}$, (5.24), and, vice versa, the new momentum variables $\boldsymbol{\pi}$ emanate, see (5.26), from the old coordinates $\mathbf{x}$.

As far as the regular gravity space approach is concerned, it can already be noted that by taking Definition 20 into account, the new coordinates $\boldsymbol{\xi}$ are related to the old momentum variables $\mathbf{p} = \nabla V(\mathbf{x})$ in the following way

$$
\boldsymbol{\xi} = -\sqrt{GM}\frac{\mathbf{p}}{\|\mathbf{p}\|^{3/2}}\,. \tag{5.27}
$$

Hence, instead of the elementary relationship (5.24) in case of F. Sansò's Legendre transformation, the situation in (5.27) is somewhat more complicated. The reason is, as discussed already in conjunction with (5.1), that by the use of the scaling factor $-\frac{\sqrt{GM}}{\|\mathbf{p}\|^{3/2}}$, a point at infinity remains fixed. Furthermore, the relation to determine the adjoint potential, as provided by (5.2), now reads as

$$
\psi = \mathbf{p}^\top \mathbf{x} - V(\mathbf{x})\,, \tag{5.28}
$$

which, as mentioned before, is formally identical to (5.25). In order to obtain the missing third equation, which will be devoted to the determination of the new momentum variables, considerations based on the subsequent lemma are made. However, it can be assumed that an altered and presumably more complex expression compared to (5.26) can be expected. This assertion is justified since a modification of the relationship, yielding the new coordinates $\boldsymbol{\xi}$ and additionally leaving the adjoint potential determination unchanged, must result in a modified computation of the new momentum variables in order to retain the identity (2.83) and thus to guarantee that the new transformation formulae form a contact transformation, which is the essential requirement for a bijective transition of geometry and gravity space.

Lemma 20 *The relationship to determine the new momentum vector* $\boldsymbol{\pi} = \nabla_\xi \psi(\boldsymbol{\xi})$ *from the old position vector* $\mathbf{x}$ *is given by*

$$
\boldsymbol{\pi} = \boldsymbol{\alpha}\mathbf{x}\,, \tag{5.29}
$$

subject to the symmetric matrix

$$
\boldsymbol{\alpha} = [\alpha_{ij}] = \left[\frac{\partial p_i}{\partial \xi_j}\right] = -\frac{GM}{\|\boldsymbol{\xi}\|^3}\left[\delta_{ij} - 3\frac{\xi_i\xi_j}{\|\boldsymbol{\xi}\|^2}\right] = -\frac{\|\mathbf{p}\|^{3/2}}{\sqrt{GM}}\left[\delta_{ij} - 3\frac{p_ip_j}{\|\mathbf{p}\|^2}\right] = \boldsymbol{\alpha}^\top\,. \tag{5.30}
$$

Proof. First of all, the inversion of (5.27) is required. That is, instead of $\boldsymbol{\xi}(\mathbf{p})$, cf. (5.27), $\mathbf{p}(\boldsymbol{\xi})$ is sought-after. For this purpose, the modulus of $\boldsymbol{\xi}$, see also (5.6), is directly derived from (5.27)

$$\|\boldsymbol{\xi}\| = \frac{\sqrt{GM}}{\sqrt{\|\mathbf{p}\|}}\,. \tag{5.31}$$

Now, solving (5.31) for $\sqrt{\|\mathbf{p}\|}$ and raising the result to the power of three yields

$$\|\mathbf{p}\|^{3/2} = \frac{GM^{3/2}}{\|\boldsymbol{\xi}\|^3}\,. \tag{5.32}$$

Thus, by substituting (5.32) in (5.27) and solving for $\mathbf{p}$, the inversion of (5.27) is deduced

$$\mathbf{p} = -GM\frac{\boldsymbol{\xi}}{\|\boldsymbol{\xi}\|^3}\,. \tag{5.33}$$

Consequently, the first order derivative of $\mathbf{p}$, given according to (5.33), with respect to $\boldsymbol{\xi}$ is derived in a straightforward manner by applying the chain rule of differential calculus

$$\begin{aligned} \nabla_\xi \mathbf{p} = \left[\frac{\partial p_i}{\partial \xi_j}\right] &= -GM\left[\frac{1}{\|\boldsymbol{\xi}\|^3}\mathbf{I} + \frac{-3}{\|\boldsymbol{\xi}\|^5}\left[\boldsymbol{\xi}\boldsymbol{\xi}^\top\right]\right] \\ &= -\frac{GM}{\|\boldsymbol{\xi}\|^3}\left[\delta_{ij} - 3\frac{\xi_i\xi_j}{\|\boldsymbol{\xi}\|^2}\right]\,. \end{aligned} \tag{5.34}$$

Next, starting from the definition $\boldsymbol{\pi} = \nabla_\xi \psi(\boldsymbol{\xi})$, a derivation is considered in the following, which is based on the substitution of $\psi(\boldsymbol{\xi})$ according to (5.28) and on the application of the chain rule of differential calculus. Thereby using index notation gives

$$\begin{aligned} \pi_i = \frac{\partial \psi}{\partial \xi_i} &= \frac{\partial}{\partial \xi_i}\left(p_k x_k - V\right) \\ &= \frac{\partial p_k}{\partial \xi_i}x_k + p_k\frac{\partial x_k}{\partial \xi_i} - \frac{\partial V}{\partial \xi_i} \\ &= \frac{\partial p_k}{\partial \xi_i}x_k + p_k\frac{\partial x_k}{\partial \xi_i} - \frac{\partial V}{\partial x_k}\frac{\partial x_k}{\partial \xi_i} \\ &= \frac{\partial p_k}{\partial \xi_i}x_k + p_k\frac{\partial x_k}{\partial \xi_i} - p_k\frac{\partial x_k}{\partial \xi_i} \\ &= \frac{\partial p_k}{\partial \xi_i}x_k\,. \end{aligned} \tag{5.35}$$

In conclusion, the last equation in (5.35), together with (5.34), as well as the insertion of (5.27) results in the following identity

$$\begin{aligned} \pi_i = \frac{\partial p_k}{\partial \xi_i}x_k &= -\frac{GM}{\|\boldsymbol{\xi}\|^3}\left[\delta_{ik} - 3\frac{\xi_i\xi_k}{\|\boldsymbol{\xi}\|^2}\right]x_k \\ &= \frac{\|\mathbf{p}\|^{3/2}}{\sqrt{GM}}\left[\delta_{ik} - 3\frac{p_i p_k}{\|\mathbf{p}\|^2}\right]x_k = \alpha_{ik}x_k\,. \qquad \diamond \end{aligned} \tag{5.36}$$

The matrix $\boldsymbol{\alpha} = \left[\frac{\partial p_i}{\partial \xi_j}\right]$ can be understood as the Jacobian matrix of a coordinate transformation of the following form

$$p_i = p_i(\xi_j)\,. \tag{5.37}$$

It is easy to assure that $\boldsymbol{\alpha}$ is a regular matrix except for $\boldsymbol{\xi} = 0$, respectively $\mathbf{p} = 0$, which can be directly seen from (5.30) and(5.27). However, this eventuality can be excluded since the underlying problem in gravity space represents an exterior problem. To obtain $\boldsymbol{\xi} = 0$ as an image point in gravity space is therefore impossible. Consequently, $\boldsymbol{\alpha}$ can be considered as nonsingular in the region of interest and thus its inverse and, as a result, the inverse of the transformation (5.37) exists. Hence, the question related to the Jacobian matrix of the associated inverse coordinate transformation

$$\xi_j = \xi_j(p_k) \tag{5.38}$$

immediately comes to mind and is therefore addressed in the context of the next lemma.

Lemma 21 *Let the symmetric matrix*

$$\boldsymbol{\gamma} = [\gamma_{kl}] = \left[\frac{\partial \xi_k}{\partial p_l}\right] = -\frac{\|\boldsymbol{\xi}\|^3}{GM}\left[\delta_{kl} - \frac{3}{2}\frac{\xi_k \xi_l}{\|\boldsymbol{\xi}\|^2}\right] = -\frac{\sqrt{GM}}{\|\mathbf{p}\|^{3/2}}\left[\delta_{kl} - \frac{3}{2}\frac{p_k p_l}{\|\mathbf{p}\|^2}\right] = \boldsymbol{\gamma}^\top \tag{5.39}$$

represent the Jacobian matrix of the coordinate transformation (5.38), then, according to Lemma 1, the condition

$$\alpha_{ik}\gamma_{kj} = \delta_{ij} \tag{5.40}$$

has to be satisfied.

Proof. Straightforward derivations demonstrate

$$\begin{aligned} \alpha_{ik}\gamma_{kj} &= \left[-\frac{\|\mathbf{p}\|^{3/2}}{\sqrt{GM}}\left[\delta_{ik} - 3\frac{p_i p_k}{\|\mathbf{p}\|^2}\right]\right]\left[-\frac{\sqrt{GM}}{\|\mathbf{p}\|^{3/2}}\left[\delta_{kj} - \frac{3}{2}\frac{p_k p_j}{\|\mathbf{p}\|^2}\right]\right] \\ &= \delta_{ik}\delta_{kj} - \frac{3}{2}\delta_{ik}\frac{p_k p_j}{\|\mathbf{p}\|^2} - 3\delta_{kj}\frac{p_i p_k}{\|\mathbf{p}\|^2} + \frac{9}{2}\frac{p_i p_k p_k p_j}{\|\mathbf{p}\|^4} \\ &= \delta_{ij} - \frac{3}{2}\frac{p_i p_j}{\|\mathbf{p}\|^2} - 3\frac{p_i p_j}{\|\mathbf{p}\|^2} + \frac{9}{2}\frac{p_i p_j}{\|\mathbf{p}\|^2} \\ &= \delta_{ij}\,. \qquad \diamondsuit \end{aligned}$$

As well as $\boldsymbol{\alpha}$, the matrix $\boldsymbol{\gamma}$ turns out to be regular except for $\mathbf{p} = 0$, which is again uncritical due to the same reasons given previous to this lemma. In addition, the recovered relationship of the functional matrices $\boldsymbol{\alpha}$ and $\boldsymbol{\gamma}$ allows for an elegant way of expressing the inverse relation of (5.29) given in the context of Lemma 20:

Lemma 22 *The inversion of (5.29) reads as*

$$\mathbf{x} = \boldsymbol{\gamma}\boldsymbol{\pi}\,. \tag{5.41}$$

Proof. The starting point is the following equation

$$\boldsymbol{\pi} = \left[\frac{\partial \mathbf{p}}{\partial \boldsymbol{\xi}}\right]\mathbf{x} \tag{5.42}$$

established in Lemma 20. Inverting (5.42) leads formally to

$$\mathbf{x} = \left[\frac{\partial \mathbf{p}}{\partial \boldsymbol{\xi}}\right]^{-1}\boldsymbol{\pi}\,,$$

which is, in view of the reasoning given above in the scope of Lemma 21, equivalent to

$$\begin{aligned} \mathbf{x} &= \left[\frac{\partial \boldsymbol{\xi}}{\partial \mathbf{p}}\right]\boldsymbol{\pi} \\ &= \boldsymbol{\gamma}\boldsymbol{\pi}\,. \qquad \diamondsuit \end{aligned}$$

At this point, it is of importance to emphasize that (5.41) constitutes the inverse relationship of (5.27), or rather (5.1). Thus, with $\boldsymbol{\gamma}$, Lemma 21, and $\boldsymbol{\pi} = \nabla_\xi \psi(\boldsymbol{\xi})$ in mind, it is possible by means of (5.41) to transform the regular gravity space coordinates ξ_i into the geometry space coordinates x_i. Next, two complementary remarks are important for a thorough understanding of the functional matrices $\boldsymbol{\alpha}$ and $\boldsymbol{\gamma}$ and the relationship of coordinates $\mathbf{x}, \boldsymbol{\xi}$ and momentum variables $\mathbf{p}, \boldsymbol{\pi}$. In this context, it is of particular importance realize that the momentum variables, just as the coordinate vectors, are independent quantities of the six-dimensional phase domain.

Remark 23 Lemma 20 constitutes the following relationship

$$\boldsymbol{\pi} = \boldsymbol{\alpha}(\mathbf{p})\mathbf{x}$$

of the momentum vector $\boldsymbol{\pi}$ and the coordinate vector $\mathbf{x}$. In fact, with the matrix $\boldsymbol{\alpha}$ being dependent only on $\mathbf{p}$, this equation represents a linear function in $\mathbf{x}$. Thus, an interpretation for the matrix $\boldsymbol{\alpha}$ can be deduced by differentiating $\boldsymbol{\pi}$ with respect to $\mathbf{x}$

$$\frac{\partial \boldsymbol{\pi}}{\partial \mathbf{x}} = \boldsymbol{\alpha}\,. \tag{5.43}$$

Remark 24 So far only the dependency of the new momentum variables $\boldsymbol{\pi}$ on the old coordinates $\mathbf{x}$ has been considered. Consequently, the relationship between the old momentum variables $\mathbf{p}$ and the new coordinates $\boldsymbol{\xi}$ is investigated next for completeness. Similarly to the modus operandi chosen to prove the statement of Lemma 20, the definition of the momentum quantity as the first order derivative of the related potential function, i.e. $\mathbf{p} = \nabla V(\mathbf{x})$, is again the starting point. Then, substitution of the potential V using Theorem 5, replacing by definition $\nabla_\xi \psi$ with $\boldsymbol{\pi}$, and application of the chain rule yields

$$\begin{aligned} p_i = \frac{\partial V}{\partial x_i} &= -\frac{\partial}{\partial x_i}\left(\frac{1}{2}\pi_k \xi_k + \psi\right) \\ &= -\frac{1}{2}\frac{\partial \pi_k}{\partial x_i}\xi_k - \frac{1}{2}\frac{\partial \xi_k}{\partial x_i}\pi_k - \frac{\partial \psi}{\partial x_i} \\ &= -\frac{1}{2}\frac{\partial \pi_k}{\partial x_i}\xi_k - \frac{1}{2}\frac{\partial \xi_k}{\partial x_i}\pi_k - \frac{\partial \psi}{\partial \xi_k}\frac{\partial \xi_k}{\partial x_i} \\ &= -\frac{1}{2}\alpha_{ik}\xi_k - \frac{1}{2}\frac{\partial \xi_k}{\partial x_i}\pi_k - \frac{\partial \xi_k}{\partial x_i}\pi_k \\ &= -\frac{1}{2}\alpha_{ik}\xi_k - \frac{3}{2}\frac{\partial \xi_k}{\partial x_i}\pi_k \,. \end{aligned}$$

Now, since the quantity $\boldsymbol{\xi}$ is independent of $\mathbf{x}$, see (5.27), it follows that the second term on the right hand side of the last relationship above becomes zero. The fact that $\mathbf{p}$ itself is a function of $\mathbf{x}$ in the space domain, i.e. $\mathbf{p} = \mathbf{p}(\mathbf{x}) = \nabla V(\mathbf{x})$, is irrelevant in the case of a contact transformation treatment in the phase domain, since the quantity $\mathbf{p}$ is considered independent in this regard, as has been discussed prior to Remark 23 and in Section 2-5. Consequently,

$$\mathbf{p} = -\frac{1}{2}\boldsymbol{\alpha}\boldsymbol{\xi} \tag{5.44}$$

is obtained for the relation of old momenta $\mathbf{p}$ and new coordinates $\boldsymbol{\xi}$. As already elaborated at the end of the previous paragraph and by comparing (5.29) and (5.44) it can be pointed out once more that the striking symmetry as in the case of F. Sansò's Legendre transformation based gravity space approach is lost.

Before the final theorem is presented, it is essential to discuss the question whether the equations

$$\boldsymbol{\xi} = -\sqrt{GM}\frac{\mathbf{p}}{\|\mathbf{p}\|^{3/2}} \tag{5.45}$$

$$\psi = \mathbf{p}^\top \mathbf{x} - V(\mathbf{x}) \tag{5.46}$$

$$\boldsymbol{\pi} = \boldsymbol{\alpha}\mathbf{x}\,, \tag{5.47}$$

which are defining W. Keller's regular gravity space approach, form a bijective transformation in general. For this purpose, the associated Jacobian matrix and its determinant are investigated. As is generally known, see also Section 2-1.2, the Jacobian is required to be non-zero. This aspect is covered in the next lemma:

Lemma 23 *Let*

$$\left[\frac{\partial(\boldsymbol{\xi}, \psi, \boldsymbol{\pi})}{\partial(\mathbf{x}, V, \mathbf{p})}\right] \tag{5.48}$$

be the functional matrix related to the regular gravity space approach described by the transformations (5.45)-(5.47), then for the determinant holds

$$\left|\left[\frac{\partial(\boldsymbol{\xi}, \psi, \boldsymbol{\pi})}{\partial(\mathbf{x}, V, \mathbf{p})}\right]\right| \neq 0\,. \tag{5.49}$$

Proof. Explicitely writing out the Jacobian matrix introduced in Lemma 23 leads to

$$\left|\left[\frac{\partial(\boldsymbol{\xi}, \psi, \boldsymbol{\pi})}{\partial(\mathbf{x}, V, \mathbf{p})}\right]\right| = \left|\begin{bmatrix} 0 & 0 & \boldsymbol{\gamma} \\ \mathbf{p}^\top & -1 & \mathbf{x}^\top \\ \boldsymbol{\alpha} & 0 & (3,3) \end{bmatrix}\right|\,,$$

with the matrix element

$$(3,3) := \frac{\partial \boldsymbol{\pi}}{\partial \mathbf{p}} = \frac{\partial(\boldsymbol{\alpha}(\mathbf{p})\mathbf{x})}{\partial \mathbf{p}} = \frac{\partial \boldsymbol{\alpha}(\mathbf{p})}{\partial \mathbf{p}}\mathbf{x} \neq 0\,.$$

As is generally known, the interchange of columns and rows leaves the value of a determinant unchanged. Thus, exchanging the second and third row as well as the first and second column yields

$$\left|\left[\frac{\partial\left(\boldsymbol{\xi},\psi,\boldsymbol{\pi}\right)}{\partial\left(\mathbf{x},V,\mathbf{p}\right)}\right]\right| = \left|\left[\begin{array}{ccc} 0 & 0 & \boldsymbol{\gamma} \\ 0 & \boldsymbol{\alpha} & (3,3) \\ -1 & \mathbf{p}^\top & \mathbf{x}^\top \end{array}\right]\right|$$

At last, *Laplace's formula* is applied to expand the determinant along the first column

$$\left|\left[\frac{\partial\left(\boldsymbol{\xi},\psi,\boldsymbol{\pi}\right)}{\partial\left(\mathbf{x},V,\mathbf{p}\right)}\right]\right| = -\left|\left[\begin{array}{cc} 0 & \boldsymbol{\gamma} \\ \boldsymbol{\alpha} & (3,3) \end{array}\right]\right| \neq 0\,. \qquad \diamondsuit$$

The conclusion that the only cofactor remaining above is unequal to zero results from the fact that both matrices $\boldsymbol{\alpha}$ and $\boldsymbol{\gamma}$ are regular as has been discussed beforehand.

At last, the following theorem identifying W. Keller's regular gravity space approach as a contact transformation completes this paragraph:

Theorem 6 *The transformations (5.45)-(5.47) are in agreement with Definition 2 and consequently constitute a contact transformation.*

Proof. Simple calculations based on the recent finding verify that (5.45)-(5.47) indeed satisfy the required condition (2.83) in order to establish a contact transformation

$$\begin{aligned} d\psi - d\xi_k\pi_k &= d(p_i x_i - V) - d\xi_k\pi_k \\ &= x_i dp_i + p_i dx_i - dV - \frac{\partial \xi_k}{\partial p_n} dp_n \alpha_{ki} x_i \\ &= x_i dp_i + p_i dx_i - dV - \gamma_{kn} dp_n \alpha_{ki} x_i \\ &= -\left(dV - dx_i p_i\right)\,. \qquad \diamondsuit \end{aligned}$$

Having discussed the transition between the ordinary geometry space and the auxiliary gravity space at length, the next question to enter deals with finding the analogon of the rotational free version of the vectorial free GBVP, Definition 13, in regular gravity space. This will be investigated in the following section.

5-2 The nonlinear GBVP in regular gravity space

The previous paragraphs have essentially accomplished two tasks. Namely, on the one hand, to introduce the new transformation formulae

$$\begin{aligned} \boldsymbol{\xi} &= -\sqrt{GM}\frac{\mathbf{p}}{\|\mathbf{p}\|^{3/2}} \\ \psi &= \mathbf{p}^\top\mathbf{x} - V(\mathbf{x}) \\ \boldsymbol{\pi} &= \boldsymbol{\alpha}\mathbf{x}\,, \end{aligned} \tag{5.50}$$

which were found to establish the new regular gravity space concept according to W. Keller in the sense of a contact transformation. And on the other hand, to investigate the underlying fundamental properties of such an approach. At the same time, some basic relationships, cf. Lemmata 18 – 22, have been already found and will now be essential to formulate the nonlinear GBVP in regular gravity space as the aim of this paragraph. Yet, two further preliminary considerations are required beforehand. That is, Laplace's equation, which represents the basic field equation in ordinary space, is transformed into the associated field equation in regular gravity space. Four consecutive lemmata address this issue. Furthermore, the boundary condition to complete the BVP is derived, where an independent verification of Theorem 5 is also performed. It is worth mentioning that the overall line of argument given here strongly resembles the one given in Section 4-4 in the context of F. Sansò's gravity space approach. A final remark comparing the GBVP in regular gravity space to the GBVP in F. Sansò's gravity space concludes this section.

To begin with, the Jacobian matrix of the forward transformation $\boldsymbol{\xi} = \boldsymbol{\xi}(\mathbf{x})$ as specified in Definition 20 and the Jacobian matrix of the corresponding inverse transformation $\mathbf{x} = \mathbf{x}(\boldsymbol{\xi})$ are related to one another, which allows the establishment of a relationship between the matrix $\mathbf{V} = \left[\frac{\partial^2 V}{\partial x_l \partial x_k}\right]$ of second order derivatives of the potential V, (2.28), and a new matrix consisting of first and second order derivatives of the adjoint potential ψ. To avoid confusion, it is again important to distinguish between considerations made in the phase domain as has been done the last section and considerations made in the more commonly used space domain. In this regard, it should be pointed out that, if not stated otherwise, the following derivations hold with respect to the three-dimensional space domain. That is, e.g., for the momentum variable $\mathbf{p}$, see (5.50) for instance, applies $\mathbf{p} = \mathbf{p}(\mathbf{x})$ in fact. To begin with, a first lemma introduces the following new quantity:

Lemma 24 *Let the triple subscripted coefficients β_{imk} be defined by*

$$\beta_{imk} := \frac{\partial \gamma_{im}}{\partial p_k} \tag{5.51}$$

then

$$\left[\frac{\partial^2 V}{\partial x_i \partial x_k}\right]^{-1} = \left[\gamma_{im}\gamma_{kl}\frac{\partial^2 \psi}{\partial \xi_m \partial \xi_l} + \beta_{imk}\frac{\partial \psi}{\partial \xi_m}\right] \tag{5.52}$$

holds.

Proof. Formally, the functional matrix given in Lemma 18 can be expressed as follows

$$\left[\frac{\partial \xi_i}{\partial x_k}\right] = \left[\frac{\partial \xi_i}{\partial p_l}\frac{\partial p_l}{\partial x_k}\right] = \left[\gamma_{il}\frac{\partial^2 V}{\partial x_l \partial x_k}\right], \tag{5.53}$$

by applying the chain rule of differential calculus and by making use of Lemma 21 together with the fact that $p_i = \frac{\partial V}{\partial x_i}$ holds. Respectively, the Jacobian matrix related to the inverse transformation, see Lemma 19, can be formally written as

$$\begin{aligned}\left[\frac{\partial x_k}{\partial \xi_j}\right] = \left[\frac{\partial(\gamma_{km}\pi_m)}{\partial \xi_j}\right] &= \left[\gamma_{km}\frac{\partial^2 \psi}{\partial \xi_m \partial \xi_j} + \frac{\partial \gamma_{km}}{\partial p_n}\frac{\partial p_n}{\partial \xi_j}\frac{\partial \psi}{\partial \xi_m}\right] \\ &= \left[\gamma_{km}\frac{\partial^2 \psi}{\partial \xi_m \partial \xi_j} + \frac{\partial \gamma_{km}}{\partial p_n}\alpha_{nj}\frac{\partial \psi}{\partial \xi_m}\right],\end{aligned} \tag{5.54}$$

again by using the chain rule as well as Lemma 20 and the familiar relation $\pi_i = \frac{\partial \psi}{\partial \xi_i}$. Now, according to Lemma 19, both matrices are inverse to each other, which yields

$$\begin{aligned}[\delta_{ij}] = \left[\frac{\partial x_i}{\partial \xi_l}\frac{\partial \xi_l}{\partial x_j}\right] &= \left[\gamma_{im}\frac{\partial^2 \psi}{\partial \xi_m \partial \xi_l} + \frac{\partial \gamma_{im}}{\partial p_n}\alpha_{nl}\frac{\partial \psi}{\partial \xi_m}\right]\left[\gamma_{lk}\frac{\partial^2 V}{\partial x_k \partial x_j}\right] \\ &= \left[\gamma_{im}\gamma_{kl}\frac{\partial^2 \psi}{\partial \xi_m \partial \xi_l} + \frac{\partial \gamma_{im}}{\partial p_k}\frac{\partial \psi}{\partial \xi_m}\right]\left[\frac{\partial^2 V}{\partial x_k \partial x_j}\right].\end{aligned}$$

Similarly to the reasoning leading to (4.11), it can be claimed that the invertability of the matrix $\mathbf{V}$ is granted. This can be directly derived from (5.53). The fact that on the one hand the transformation $\boldsymbol{\xi} = \boldsymbol{\xi}(\mathbf{x})$ is unique by definition, and that on the other hand the matrix $\boldsymbol{\gamma}$ is regular as discussed before, allows the conclusion of a regular matrix $\mathbf{V}$. Thus, it follows

$$\left[\frac{\partial^2 V}{\partial x_j \partial x_k}\right]^{-1} = \left[\gamma_{jm}\gamma_{kl}\frac{\partial^2 \psi}{\partial \xi_m \partial \xi_l} + \beta_{jmk}\frac{\partial \psi}{\partial \xi_m}\right]. \qquad \diamondsuit$$

During the course of the ongoing study, e.g. in the context of the subsequent linearization process presented next, an explicit knowledge of the functions $\beta_{imk} := \frac{\partial \gamma_{im}}{\partial p_k}$ will be of importance. Hence, this requirement is met by the next lemma:

Lemma 25 *Evaluation of the coefficients β_{imk} leads to*

$$\beta_{imk} = -\frac{3}{2}\frac{\|\boldsymbol{\xi}\|^4}{GM^2}\left(\delta_{im}\xi_k + \delta_{mk}\xi_i + \delta_{ik}\xi_m - \frac{7}{2}\frac{\xi_i\xi_m\xi_k}{\|\boldsymbol{\xi}\|^2}\right). \tag{5.55}$$

Proof. Starting from the definition of the coefficients β_{imk} as provided by Lemma (24) the following relation,

$$\begin{aligned}\beta_{imk} &= \frac{\partial\gamma_{im}}{\partial p_k} \\ &= \frac{\partial\gamma_{im}}{\partial\xi_l}\frac{\partial\xi_l}{\partial p_k} \\ &= \frac{\partial\gamma_{im}}{\partial\xi_l}\gamma_{lk}\,,\end{aligned} \tag{5.56}$$

can easily be deduced from application of the chain rule of differential calculus. Then, in order to determine the first term on the right hand side of (5.56), the functional matrix $\boldsymbol{\gamma}$ is recalled from Lemma 21 and its differentiation with respect to $\boldsymbol{\xi}$

$$\frac{\partial\gamma_{im}}{\partial\xi_l} = \frac{\partial}{\partial\xi_l}\left(-\frac{\|\boldsymbol{\xi}\|^3}{GM}\left[\delta_{im} - \frac{3}{2}\frac{\xi_i\xi_m}{\|\boldsymbol{\xi}\|^2}\right]\right) \tag{5.57}$$

is obtained successively as follows. At first, differentiation of to the cubed magnitude of $\boldsymbol{\xi}$ is considered

$$\frac{\partial}{\partial\xi_l}\left(\|\boldsymbol{\xi}\|^3\right) = 3\|\boldsymbol{\xi}\|\xi_l\,. \tag{5.58}$$

Next, differentiating the remaining part according to the product rule of differential calculus yields

$$\frac{\partial}{\partial\xi_l}\left(\|\boldsymbol{\xi}\|\xi_i\xi_m\right) = \frac{\xi_l\xi_i\xi_m}{\|\boldsymbol{\xi}\|} + \|\boldsymbol{\xi}\|\delta_{il}\xi_m + \|\boldsymbol{\xi}\|\delta_{ml}\xi_i\,. \tag{5.59}$$

Combining (5.58) and (5.59) leads to the derivative required by (5.57)

$$\frac{\partial\gamma_{im}}{\partial\xi_l} = -\frac{1}{GM}\left(3\|\boldsymbol{\xi}\|\delta_{im}\xi_l - \frac{3}{2\|\boldsymbol{\xi}\|}\left(\xi_i\xi_l\xi_m + \|\boldsymbol{\xi}\|^2\delta_{il}\xi_m + \|\boldsymbol{\xi}\|^2\delta_{ml}\xi_i\right)\right)\,. \tag{5.60}$$

Finally, evaluating the expression on the right hand side of (5.56), i.e. multiplying the relationship given in (5.57) by the matrix $\boldsymbol{\gamma}$, corroborates the hypothesis of Lemma 25

$$\begin{aligned}\frac{\partial\gamma_{im}}{\partial\xi_l}\gamma_{lk} &= \frac{\|\boldsymbol{\xi}\|^3}{GM^2}\left(3\|\boldsymbol{\xi}\|\delta_{im}\xi_k - \frac{9}{2}\|\boldsymbol{\xi}\|\delta_{im}\xi_k - \frac{3}{2\|\boldsymbol{\xi}\|}\xi_i\xi_m\xi_k + \frac{9}{4\|\boldsymbol{\xi}\|}\xi_i\xi_m\xi_k\right. \\ &\qquad \left. -\frac{3}{2}\|\boldsymbol{\xi}\|\delta_{ik}\xi_m + \frac{9}{4\|\boldsymbol{\xi}\|}\xi_i\xi_m\xi_k - \frac{3}{2}\|\boldsymbol{\xi}\|\delta_{mk}\xi_i + \frac{9}{4\|\boldsymbol{\xi}\|}\xi_i\xi_m\xi_k\right) \\ &= \frac{\|\boldsymbol{\xi}\|^3}{GM^2}\left(-\frac{3}{2}\|\boldsymbol{\xi}\|\delta_{im}\xi_k - \frac{3}{2}\|\boldsymbol{\xi}\|\delta_{mk}\xi_i - \frac{3}{2}\|\boldsymbol{\xi}\|\delta_{ik}\xi_m + \frac{21}{4\|\boldsymbol{\xi}\|}\xi_i\xi_m\xi_k\right) \\ &= -\frac{3}{2}\frac{\|\boldsymbol{\xi}\|^4}{GM^2}\left(\delta_{im}\xi_k + \delta_{mk}\xi_i + \delta_{ik}\xi_m - \frac{7}{2}\frac{\xi_i\xi_m\xi_k}{\|\boldsymbol{\xi}\|^2}\right) \\ &= \beta_{imk}\,. \qquad \diamondsuit\end{aligned}$$

As mentioned earlier, the main intention of this paragraph is to find the underlying field equation of the GBVP in regular gravity space. Now, by means of Lemma 24, the matrix $\mathbf{V}$ can be related to a new matrix $\boldsymbol{\Phi}$. Consequently, the former Laplace's equation is transformed into a new partial differential equation, which forms the corresponding field equation in regular gravity space. This is the basis for the next two lemmata:

Lemma 26 *Let* $\boldsymbol{\Phi}$ *denote the following matrix*

$$\boldsymbol{\Phi} = [\Phi_{ik}] = \left[\gamma_{im}\gamma_{kl}\frac{\partial^2\psi}{\partial\xi_m\partial\xi_l} + \beta_{imk}\frac{\partial\psi}{\partial\xi_m}\right]\,, \tag{5.61}$$

then

$$\operatorname{tr}\boldsymbol{\Phi}^{-1} = 0 \tag{5.62}$$

holds for the nonlinear field equation in regular gravity space.

Proof. Bearing (2.28) in mind, the matrix identity

$$\mathbf{V} = \boldsymbol{\Phi}^{-1} \tag{5.63}$$

results directly from Lemma 24. As far as the field equation is concerned, which in the case of a consideration in ordinary space is given by Laplace's equation, (2.31), the following conclusion with regard to the corresponding partial differential equation for the adjoint potential can be drawn

$$\Delta V = \operatorname{tr} \mathbf{V} = \operatorname{tr} \mathbf{\Phi}^{-1} = 0\,. \qquad \diamondsuit \tag{5.64}$$

Analogously to the methodical procedure that has already been adopted in the course of treating F. Sansò's gravity space approach, the field equation established by Lemma 26 can be further processed:

Lemma 27 *The field equation in regular gravity space, (5.62), can be transformed into the following expression*

$$(\operatorname{tr} \mathbf{\Phi})^2 - \operatorname{tr} \mathbf{\Phi}^2 = 0\,. \tag{5.65}$$

Proof. The line of argument is essentially the same as given in the context of Lemma 11 and is therefore not repeated here again.

Besides the differential equation, which the adjoint potential ψ has to fulfill in the exterior of Σ, the behaviour of ψ on Σ itself, i.e. the boundary condition, is of fundamental interest. On this account, the boundary data $v = V\big|_\sigma$ are related by means of (5.45)-(5.47) to the associated quantities defined in regular gravity space. That is, solving (5.46) for V and restricting the involved quantities onto the corresponding boundary surface yields

$$V\big|_\sigma = \left(\mathbf{p}^\top \mathbf{x}\right)\Big|_\sigma - \psi\big|_\Sigma\,. \tag{5.66}$$

Then, replacing the momentum variable $\mathbf{p}$ in (5.66) according to (5.33) and the position vector $\mathbf{x}$ according to Lemma 22, thereby taking (5.39) and the relation $\boldsymbol{\pi} = \nabla_\xi \psi$ into account, leads to

$$\begin{aligned} V\big|_\sigma &= \left(-GM\frac{\boldsymbol{\xi}^\top}{\|\boldsymbol{\xi}\|^3}\gamma\nabla_\xi\psi\right)\Bigg|_\Sigma - \psi\big|_\Sigma \\ &= \left(\boldsymbol{\xi}^\top\left[\mathbf{I} - \frac{3}{2}\frac{\left[\boldsymbol{\xi}\boldsymbol{\xi}^\top\right]}{\|\boldsymbol{\xi}\|^2}\right]\nabla_\xi\psi - \psi\right)\Bigg|_\Sigma \\ &= \left(\left(\boldsymbol{\xi}^\top - \frac{3}{2}\frac{\boldsymbol{\xi}^\top\boldsymbol{\xi}}{\|\boldsymbol{\xi}\|^2}\boldsymbol{\xi}^\top\right)\nabla_\xi\psi - \psi\right)\Bigg|_\Sigma \\ &= -\left(\frac{1}{2}\boldsymbol{\xi}^\top\nabla_\xi\psi + \psi\right)\Bigg|_\Sigma\,. \end{aligned} \tag{5.67}$$

As already indicated in the beginning of this paragraph, Lemma 10 & 11, given in the context of F. Sansò's approach, and Lemma 26 & 27 are closely affiliated with each other. Furthermore, as far as the boundary conditions are concerned, it is safe to state that (4.33) and (5.67) exhibit a certain similarity. As a consequence, the resulting BVP in regular gravity space exhibits the same formal structure as the BVP in gravity space and reads as follows:

Definition 22 *The geodetic boundary value problem in regular gravity space constitutes the following problem: the data, i.e. gravitational potential values, are now given at the known surface* Σ

$$v : \Sigma \to \mathbb{R}$$

and needed to be found is a real function $\psi(\boldsymbol{\xi}) : ext\ \Sigma \to \mathbb{R}$

$$\begin{aligned} (\operatorname{tr} \mathbf{\Phi})^2 - \operatorname{tr} \mathbf{\Phi}^2 &= 0\,, \quad \boldsymbol{\xi} \in ext\ \Sigma && (5.68) \\ -\left(\frac{1}{2}\boldsymbol{\xi}^\top\nabla_\xi\psi + \psi\right)\Bigg|_\Sigma &= v\,, && (5.69) \end{aligned}$$

which is the solution of the second-order partial differential equation (5.68) under the boundary condition (5.69).

Comparably to Definition 17, (5.68) and (5.69) establish a nonlinear oblique BVP. Except for the coefficients γ_{ij} and β_{imj} associated with $\mathbf{\Phi}$, the structure of the underlying field equation is similar to a Monge-Ampère type of differential equation. The boundary condition (5.69) is linear and results from the reasoning given in (5.67), which is in agreement with Theorem 5. Together, (5.68) and (5.69) establish the GBVP in regular gravity space, a BVP with a *fixed* boundary, since in contrast to the Earth's surface σ, the boundary surface Σ is known. Similarly to Definition 15, the position vector $\boldsymbol{\xi}\big|_\Sigma$ in gravity space describes the known boundary surface Σ as the image of the Earth's surface σ, however this time under the regular gravity space mapping (5.1)

$$\boldsymbol{\xi}\big|_\Sigma = -\sqrt{GM}\frac{\nabla V(\mathbf{x}\big|_\sigma)}{\|\nabla V(\mathbf{x}\big|_\sigma)\|^{3/2}}\,.$$

Alternatively, by taking Lemma 17 and Definition 21 into account, the surface Σ can be derived in such a way that the condition

$$\nabla V_0(\boldsymbol{\xi}\big|_\Sigma) = \nabla V(\mathbf{x}\big|_\sigma)$$

is fulfilled. Finally, after having obtained the solution for $\psi(\boldsymbol{\xi})$, the Earth's surface σ, according to Lemma 22, is given by

$$\begin{aligned}\mathbf{x}\big|_\sigma &= \gamma(\boldsymbol{\xi}\big|_\Sigma)\boldsymbol{\pi}(\boldsymbol{\xi}\big|_\Sigma)\\ &= \gamma(\boldsymbol{\xi}\big|_\Sigma)\nabla_\xi\psi(\boldsymbol{\xi}\big|_\Sigma)\,.\end{aligned} \tag{5.70}$$

Remark 25 At last, it is valuable to comment on the difference of the GBVP in gravity space and the GBVP in regular gravity space. As mentioned above, the associated BVPs were found to be of a similar mathematical structure. However, there exists a fine difference between the gravity space and the regular gravity space concept. In fact, both problems differ in the elements of the matrices $\boldsymbol{\Psi}$ and $\boldsymbol{\Phi}$, respectively. The former matrix $\boldsymbol{\Psi}$ is only composed of the second order derivatives of the adjoint potential ψ, whereas the last-mentioned matrix $\boldsymbol{\Phi}$ is based on a linear combination of first and second order derivatives of the adjoint potential ψ. In both cases the BVP feature linear, oblique boundary conditions.

5-3 Linearization of the GBVP in regular gravity space

As accomplished already in Section 3-4 and Section 4-5, the commonly practiced linearization step of a nonlinear problem, as established for example in the form of the nonlinear fixed GBVP in regular gravity space at the end of the last section, is achieved once more in the familiar way. Strictly speaking, the procedure to linearize the nonlinear fixed GBVP in gravity space, Section 4-5, is repeated to some extent in an analogous manner. That is, the adjoint potential (5.2) is approximated by an adequate *adjoint normal potential*, which is shown to satisfy the partial differential equation (5.65) valid for the nonlinear BVP in regular gravity space. Eventually, by formally substituting the sum of adjoint normal potential and *adjoint disturbing potential* instead of the true adjoint potential into the nonlinear field equation (5.65), thereby neglecting all terms of order two and higher, the corresponding partial differential equation in linear approximation for the *adjoint disturbing potential* is found.
In more detail, within the scope of the first theorem the adjoint normal potential ψ_0, which corresponds to the normal gravitational potential $V_0 = \frac{GM}{\|\mathbf{x}\|}$, is established. As previously explained, ψ_0 represents a first order approximation of the true adjoint potential ψ. Then, the matrix $\boldsymbol{\Phi}_0$ of first and second order derivatives of ψ_0, cf. (5.61), is derived explicitly by means of a complementing lemma. Furthermore, a second lemma confirms that besides the adjoint potential ψ, the adjoint normal potential ψ_0 also satisfies the field equation of regular gravity space as specified in Lemma 27. Next, the nonlinear partial differential equation for ψ is linearized in a second lemma by means of splitting the adjoint potential in (5.65) into its reference and its disturbing part and by restricting the resulting equation to the linear terms. A second theorem claims that in regular gravity space the *adjoint disturbing potential* this time indeed satisfies Laplace's equation. These considerations concerning the field equation for the adjoint disturbing potential are followed by investigations on the corresponding boundary condition. Finally, the linearized version of the BVP (5.68)-(5.69), given at the end of Section 5-2, concludes this section.

In the manner described before in Theorem 4, the adjoint normal potential, which in the framework of the regular gravity space approach also results from the isotropic normal potential (2.40), is determined in the following theorem:

Theorem 7 *If*

$$V_0(\mathbf{x}) = \frac{GM}{\|\mathbf{x}\|} \tag{5.71}$$

holds for the gravitational normal potential, then the corresponding adjoint normal potential reads as

$$\psi_0 = -2\frac{GM}{\|\boldsymbol{\xi}\|} . \tag{5.72}$$

Proof. At first, (5.71) is substituted into (5.2) to obtain the following equation for the adjoint normal potential

$$\psi_0 = -\frac{GM}{\|\mathbf{x}\|^3}\mathbf{x}^\top\mathbf{x} - \frac{GM}{\|\mathbf{x}\|} = -2\frac{GM}{\|\mathbf{x}\|} , \tag{5.73}$$

which is the same relation as obtained in (4.40). Accordingly, insertion of (5.71) into the expression (5.6) found for the modulus of $\boldsymbol{\xi}$ in Section 5-1.1 yields

$$\|\boldsymbol{\xi}\| = \frac{\sqrt{GM}}{\sqrt{\|\nabla V_0(\mathbf{x})\|}} = \frac{\sqrt{GM}}{\sqrt{\frac{GM}{\|\mathbf{x}\|^2}}} = \|\mathbf{x}\| ,$$

which is naturally in agreement with Lemma 16. Finally, replacing $\|\mathbf{x}\|$ in (5.73) by $\|\boldsymbol{\xi}\|$ leads to

$$\psi_0 = -2\frac{GM}{\|\boldsymbol{\xi}\|} . \qquad \diamondsuit$$

In order to prove that the adjoint normal potential complies with the partial differential equation (5.65), the following preparational considerations made in the next lemma are necessary:

Lemma 28 *Let $\boldsymbol{\Phi}_0$ denote the matrix*

$$\boldsymbol{\Phi}_0 = \left[\Phi^0_{ik}\right] = \left[\gamma_{im}\gamma_{kl}\frac{\partial^2\psi_0}{\partial\xi_m\partial\xi_l} + \beta_{imk}\frac{\partial\psi_0}{\partial\xi_m}\right] , \tag{5.74}$$

then in consideration of the functional matrix $\boldsymbol{\gamma}$, see Lemma 21, the following simple relationship is true

$$\boldsymbol{\Phi}_0 = \boldsymbol{\gamma} . \tag{5.75}$$

Proof. The explicit evaluation of (5.74), subject to the adjoint normal potential ψ_0 given according to (5.72), is split up in three steps. First, the second order derivatives of ψ_0 are determined

$$\begin{aligned}
\frac{\partial^2\psi_0}{\partial\xi_i\partial\xi_j} = -2GM\frac{\partial}{\partial\xi_j}\left(\frac{\partial}{\partial\xi_i}\left(\frac{1}{\|\boldsymbol{\xi}\|}\right)\right) &= 2GM\frac{\partial}{\partial\xi_j}\left(\frac{\xi_i}{\|\boldsymbol{\xi}\|^3}\right)\\
&= 2GM\left[\frac{\delta_{ij}\|\boldsymbol{\xi}\|^3 - 3\xi_i\xi_j\|\boldsymbol{\xi}\|}{\|\boldsymbol{\xi}\|^6}\right]\\
&= 2\frac{GM}{\|\boldsymbol{\xi}\|^3}\left[\delta_{ij} - 3\frac{\xi_i\xi_j}{\|\boldsymbol{\xi}\|^2}\right]\\
&= -2\alpha_{ij} .
\end{aligned} \tag{5.76}$$

Then, in order to receive the complete first term in (5.74), the following considerations based on (5.76) apply

$$\begin{aligned}
\gamma_{mi}\gamma_{nj}\frac{\partial^2\psi_0}{\partial\xi_i\partial\xi_j} &= -2\gamma_{mi}\gamma_{nj}\alpha_{ij}\\
&= -2\delta_{mj}\gamma_{nj}\\
&= -2\gamma_{mn} .
\end{aligned} \tag{5.77}$$

Now, the second part of (5.74) is obtained by means of straightforward computations

$$\begin{aligned}
\beta_{imj}\frac{\partial\psi_0}{\partial\xi_m} &= -\frac{3}{2}\frac{\|\boldsymbol{\xi}\|^4}{GM^2}\left(\delta_{mi}\xi_j + \delta_{mj}\xi_i + \delta_{ij}\xi_m - \frac{7}{2}\frac{\xi_i\xi_m\xi_j}{\|\boldsymbol{\xi}\|^2}\right)\left(2\frac{GM\xi_m}{\|\boldsymbol{\xi}\|^3}\right)\\
&= -3\frac{\|\boldsymbol{\xi}\|}{GM}\left[\xi_i\xi_j + \xi_j\xi_i + \delta_{ij}\|\boldsymbol{\xi}\|^2 - \frac{7}{2}\xi_i\xi_j\right]\\
&= -3\frac{\|\boldsymbol{\xi}\|^3}{GM}\left[\delta_{ij} - \frac{3}{2}\frac{\xi_i\xi_j}{\|\boldsymbol{\xi}\|^2}\right]\\
&= 3\gamma_{ij} .
\end{aligned} \tag{5.78}$$

Finally, adding up (5.77) and (5.78) yields

$$\begin{aligned} \Phi^0_{ij} &= \gamma_{im}\gamma_{jn}\frac{\partial^2\psi_0}{\partial\xi_m\partial\xi_n} + \beta_{imj}\frac{\partial\psi_0}{\partial\xi_m} \\ &= -2\gamma_{ij} + 3\gamma_{ij} = \gamma_{ij}\,. \qquad \diamondsuit \end{aligned}$$

Next, it is demonstrated that the adjoint normal potential ψ_0, cf. Theorem 7, indeed fulfills the field equation in regular gravity space as given in Lemma 27:

Lemma 29 *The adjoint normal potential ψ_0, (5.72), satisfies the following partial differential equation*

$$(\mathrm{tr}\,\mathbf{\Phi}_0)^2 - \mathrm{tr}\,\mathbf{\Phi}_0^2 = 0\,. \tag{5.79}$$

Proof. By taking into account the relationship $\mathbf{\Phi}_0 = \boldsymbol{\gamma}$ according to Lemma 28, the first term of (5.79) is evaluated in an analogous way adopted before in order to prove Lemma 13

$$\begin{aligned} (\mathrm{tr}\,\mathbf{\Phi}_0)^2 = (\mathrm{tr}\,\boldsymbol{\gamma})^2 = \left(\mathrm{tr}\left[\frac{-\|\boldsymbol{\xi}\|^3}{GM}\left[\delta_{ij} - \frac{3}{2}\frac{\xi_i\xi_j}{\|\boldsymbol{\xi}\|^2}\right]\right]\right)^2 &= \left(\frac{-\|\boldsymbol{\xi}\|^3}{GM}\left(\delta_{ii} - \frac{3}{2}\frac{\xi_i\xi_i}{\|\boldsymbol{\xi}\|^2}\right)\right)^2 \\ &= \left(\frac{-\|\boldsymbol{\xi}\|^3}{GM}\left(3 - \frac{3}{2}\right)\right)^2 \\ &= \frac{9}{4}\frac{\|\boldsymbol{\xi}\|^6}{GM^2}\,. \end{aligned} \tag{5.80}$$

Consequently, the second term of (5.79) gives

$$\begin{aligned} \mathrm{tr}\,\mathbf{\Phi}_0^2 = \mathrm{tr}\,\boldsymbol{\gamma}^2 = \mathrm{tr}\left[\frac{-\|\boldsymbol{\xi}\|^3}{GM}\left[\delta_{ij} - \frac{3}{2}\frac{\xi_i\xi_j}{\|\boldsymbol{\xi}\|^2}\right]\right]^2 &= \mathrm{tr}\left[\frac{\|\boldsymbol{\xi}\|^6}{GM^2}\left[\delta_{ij} - \frac{3}{2}\frac{\xi_i\xi_j}{\|\boldsymbol{\xi}\|^2}\right]\left[\delta_{jk} - \frac{3}{2}\frac{\xi_j\xi_k}{\|\boldsymbol{\xi}\|^2}\right]\right] \\ &= \frac{\|\boldsymbol{\xi}\|^6}{GM^2}\,\mathrm{tr}\left[\delta_{ik} - \frac{3}{2}\frac{\xi_i\xi_k}{\|\boldsymbol{\xi}\|^2} - \frac{3}{2}\frac{\xi_i\xi_k}{\|\boldsymbol{\xi}\|^2} + \frac{9}{4}\frac{\xi_i\xi_k\|\boldsymbol{\xi}\|^2}{\|\boldsymbol{\xi}\|^4}\right] \\ &= \frac{\|\boldsymbol{\xi}\|^6}{GM^2}\left(\delta_{ii} - \frac{3}{2}\frac{\xi_i\xi_i}{\|\boldsymbol{\xi}\|^2} - \frac{3}{2}\frac{\xi_i\xi_i}{\|\boldsymbol{\xi}\|^2} + \frac{9}{4}\frac{\xi_i\xi_i}{\|\boldsymbol{\xi}\|^2}\right) \\ &= \frac{\|\boldsymbol{\xi}\|^6}{GM^2}\left(3 - \frac{3}{2} - \frac{3}{2} + \frac{9}{4}\right) \\ &= \frac{9}{4}\frac{\|\boldsymbol{\xi}\|^6}{GM^2}\,. \end{aligned} \tag{5.81}$$

Therefore, by means of (5.80) and (5.81), it follows directly that

$$(\mathrm{tr}\,\mathbf{\Phi}_0)^2 - \mathrm{tr}\,\mathbf{\Phi}_0^2 = \frac{9}{4}\frac{\|\boldsymbol{\xi}\|^6}{GM^2} - \frac{9}{4}\frac{\|\boldsymbol{\xi}\|^6}{GM^2} = 0\,. \qquad \diamondsuit$$

Following the reasoning already given in Section 4-5 the unknown adjoint potential $\psi(\boldsymbol{\xi})$ is again split into its known normal part $\psi_0(\boldsymbol{\xi})$ and its disturbing part $\delta\psi(\boldsymbol{\xi})$

$$\psi = \psi_0 + \delta\psi\,. \tag{5.82}$$

Accordingly, the procedure of separating into a reference and a disturbing part is repeated in a similar manner for the corresponding functional matrix $\mathbf{\Phi}$

$$\mathbf{\Phi} = \mathbf{\Phi}_0 + \delta\mathbf{\Phi}\,, \tag{5.83}$$

see also Lemma 26 and Lemma 28, subject to the matrix $\delta\mathbf{\Phi}$

$$\delta\mathbf{\Phi} = [\delta\Phi_{ik}] = \left[\gamma_{im}\gamma_{kl}\frac{\partial^2\delta\psi}{\partial\xi_m\partial\xi_l} + \beta_{imk}\frac{\partial\delta\psi}{\partial\xi_m}\right]\,. \tag{5.84}$$

Hence, in order to obtain a linearized form for the nonlinear GBVP in regular gravity space, established at the end of Section 5-2, the following lemma is given:

Lemma 30 *The nonlinear field equation (5.65) can be transformed into the following partial differential equation*

$$\mathrm{tr}\,\mathbf{\Phi}_0 \mathrm{tr}\,\delta\mathbf{\Phi} - \mathrm{tr}\,[\mathbf{\Phi}_0 \delta\mathbf{\Phi}] = 0 \tag{5.85}$$

by taking advantage of Lemma 29 and by neglecting terms of order $\mathrm{O}\big(\delta\mathbf{\Phi}^2\big)$.

Proof. Similarly to (4.51), insertion of (5.83) into (5.65) produces

$$\begin{aligned} 0 = (\mathrm{tr}\,\mathbf{\Phi})^2 - \mathrm{tr}\,\mathbf{\Phi}^2 &= (\mathrm{tr}\,[\mathbf{\Phi}_0 + \delta\mathbf{\Phi}])^2 - \mathrm{tr}\,[\mathbf{\Phi}_0 + \delta\mathbf{\Phi}]^2 \\ &= (\mathrm{tr}\,\mathbf{\Phi}_0)^2 + 2\mathrm{tr}\,\mathbf{\Phi}_0 \mathrm{tr}\,\delta\mathbf{\Phi} + (\mathrm{tr}\,\delta\mathbf{\Phi})^2 - \mathrm{tr}\,\mathbf{\Phi}_0^2 - 2\mathrm{tr}\,[\mathbf{\Phi}_0\delta\mathbf{\Phi}] - \mathrm{tr}\,\delta\mathbf{\Phi}^2 \\ &= 2\mathrm{tr}\,\mathbf{\Phi}_0 \mathrm{tr}\,\delta\mathbf{\Phi} - 2\mathrm{tr}\,[\mathbf{\Phi}_0\delta\mathbf{\Phi}] + (\mathrm{tr}\,\mathbf{\Phi}_0)^2 - \mathrm{tr}\,\mathbf{\Phi}_0^2 + \mathrm{O}\big(\delta\mathbf{\Phi}^2\big)\,. \end{aligned} \tag{5.86}$$

Thus, considering (5.86) and by taking Lemma 29 into account, (5.65) becomes

$$\mathrm{tr}\,\mathbf{\Phi}_0 \mathrm{tr}\,\delta\mathbf{\Phi} - \mathrm{tr}\,[\mathbf{\Phi}_0 \delta\mathbf{\Phi}] = 0\,. \qquad \diamondsuit$$

Now, it can be demonstrated based on the last lemma that in regular gravity space, the adjoint disturbing potential $\delta\psi$ satisfies Laplace's equation which, as discussed in Remark 20, is in contrast to the gravity space approach introduced by F. Sansò. Thus, elaborating the harmonic character of $\delta\psi$ is the subject matter of the following theorem:

Theorem 8 *In regular gravity space, the field equation for the adjoint disturbing potential $\delta\psi$ reads as follows*

$$\Delta\delta\psi = 0\,. \tag{5.87}$$

Proof. In order to verify that the adjoint disturbing potential $\delta\psi$ meets Laplace's equation, it is necessary to consider the following three basic principles. In the first place, $\mathrm{tr}\,\mathbf{\Phi}_0$ or rather $\mathrm{tr}\,\boldsymbol{\gamma}$, see Lemma 28, is evaluated

$$\mathrm{tr}\,[\gamma_{ij}] = \mathrm{tr}\left[-\frac{\|\boldsymbol{\xi}\|^3}{GM}\left[\delta_{ij} - \frac{3}{2}\frac{\xi_i\xi_j}{\|\boldsymbol{\xi}\|^2}\right]\right] = -\frac{\|\boldsymbol{\xi}\|^3}{GM}\left(\delta_{ii} - \frac{3}{2}\frac{\xi_i\xi_i}{\|\boldsymbol{\xi}\|^2}\right) = -\frac{\|\boldsymbol{\xi}\|^3}{GM}\left(3 - \frac{3}{2}\right) = -\frac{3}{2}\frac{\|\boldsymbol{\xi}\|^3}{GM}\,. \tag{5.88}$$

Secondly, the matrix product

$$\begin{aligned} \gamma_{ij}\gamma_{jm} &= -\frac{\|\boldsymbol{\xi}\|^6}{GM^2}\left[\left[\delta_{ij} - \frac{3}{2}\frac{\xi_i\xi_j}{\|\boldsymbol{\xi}\|^2}\right]\left[\delta_{jm} - \frac{3}{2}\frac{\xi_j\xi_m}{\|\boldsymbol{\xi}\|^2}\right]\right] = \frac{\|\boldsymbol{\xi}\|^6}{GM^2}\left[\delta_{im} - \frac{3}{2}\frac{\xi_i\xi_m}{\|\boldsymbol{\xi}\|^2} - \frac{3}{2}\frac{\xi_i\xi_m}{\|\boldsymbol{\xi}\|^2} + \frac{9}{4}\frac{\xi_i\xi_m}{\|\boldsymbol{\xi}\|^2}\frac{\xi_j\xi_j}{\|\boldsymbol{\xi}\|^2}\right] \\ &= \frac{\|\boldsymbol{\xi}\|^6}{GM^2}\left[\delta_{im} - \frac{3}{4}\frac{\xi_i\xi_m}{\|\boldsymbol{\xi}\|^2}\right] \end{aligned} \tag{5.89}$$

will be of interest. Thirdly, the subsequent relation is required in the further course

$$\begin{aligned} \gamma_{ij}\beta_{jmk} &= \frac{3}{2}\frac{\|\boldsymbol{\xi}\|^7}{GM^3}\left(\delta_{ij} - \frac{3}{2}\frac{\xi_i\xi_j}{\|\boldsymbol{\xi}\|^2}\right)\left(\delta_{jm}\xi_k + \delta_{mk}\xi_j + \delta_{jk}\xi_m - \frac{7}{2}\frac{\xi_j\xi_k\xi_m}{\|\boldsymbol{\xi}\|^2}\right) \\ &= \frac{3}{2}\frac{\|\boldsymbol{\xi}\|^7}{GM^3}\left(\delta_{im}\xi_k + \delta_{mk}\xi_i + \delta_{ik}\xi_m - \frac{7}{2}\frac{\xi_i\xi_k\xi_m}{\|\boldsymbol{\xi}\|^2} - \frac{3}{2}\frac{\xi_i\xi_k\xi_m}{\|\boldsymbol{\xi}\|^2} - \frac{3}{2}\delta_{mk}\frac{\xi_i\xi_j^2}{\|\boldsymbol{\xi}\|^2} - \frac{3}{2}\frac{\xi_i\xi_k\xi_m}{\|\boldsymbol{\xi}\|^2} + \frac{21}{4}\frac{\xi_i\xi_k\xi_m}{\|\boldsymbol{\xi}\|^2}\frac{\xi_j^2}{\|\boldsymbol{\xi}\|^2}\right) \\ &= \frac{3}{2}\frac{\|\boldsymbol{\xi}\|^7}{GM^3}\left(\delta_{im}\xi_k - \frac{1}{2}\delta_{mk}\xi_i + \delta_{ik}\xi_m - \frac{5}{4}\frac{\xi_i\xi_k\xi_m}{\|\boldsymbol{\xi}\|^2}\right)\,. \end{aligned} \tag{5.90}$$

Now, by expressing (5.85) in index notation, that is

$$\mathrm{tr}\,\left[\Phi^0_{ij}\right]\mathrm{tr}\,[\delta\Phi_{ij}] - \mathrm{tr}\,\left[\left[\Phi^0_{ij}\right][\delta\Phi_{jk}]\right] = 0\,,$$

and by making use of Lemma 28 and (5.84),

$$\mathrm{tr}\,[\gamma_{ij}]\,\mathrm{tr}\left[\gamma_{im}\gamma_{jn}\frac{\partial^2\delta\psi}{\partial\xi_m\partial\xi_n} + \beta_{imj}\frac{\partial\delta\psi}{\partial\xi_m}\right] - \mathrm{tr}\left[[\gamma_{ij}]\left[\gamma_{jm}\gamma_{kn}\frac{\partial^2\delta\psi}{\partial\xi_m\partial\xi_n} + \beta_{jmk}\frac{\partial\delta\psi}{\partial\xi_m}\right]\right] = 0 \tag{5.91}$$

is obtained. Next, (5.88) is used to rewrite the first term of the previous equation. In addition, with respect to the second term of (5.91), adopting (5.89) and (5.90) leads to

$$\begin{aligned} -\frac{3}{2}\frac{\|\boldsymbol{\xi}\|^3}{GM}\mathrm{tr}\left[\gamma_{im}\gamma_{jn}\frac{\partial^2\delta\psi}{\partial\xi_m\partial\xi_n} + \beta_{imj}\frac{\partial\delta\psi}{\partial\xi_m}\right] - \mathrm{tr}\left[\frac{-\|\boldsymbol{\xi}\|^9}{GM^3}\left[\delta_{im} - \frac{3}{4}\frac{\xi_i\xi_m}{\|\boldsymbol{\xi}\|^2}\right]\left[\delta_{kn} - \frac{3}{2}\frac{\xi_k\xi_n}{\|\boldsymbol{\xi}\|^2}\right]\frac{\partial^2\delta\psi}{\partial\xi_m\partial\xi_n} + \right. \\ \left. + \frac{3}{2}\frac{\|\boldsymbol{\xi}\|^7}{GM^3}\left(\delta_{im}\xi_k - \frac{1}{2}\delta_{mk}\xi_i + \delta_{ik}\xi_m - \frac{5}{4}\frac{\xi_i\xi_k\xi_m}{\|\boldsymbol{\xi}\|^2}\right)\frac{\partial\delta\psi}{\partial\xi_m}\right] &= 0\,. \end{aligned} \tag{5.92}$$

Then, evaluation of both trace operators in (5.92) and substitution of β_{imj} according to Lemma 25 yields

$$\begin{aligned}
-\frac{3}{2}\frac{\|\boldsymbol{\xi}\|^9}{GM^3}\left(\delta_{im}-\frac{3}{2}\frac{\xi_i\xi_m}{\|\boldsymbol{\xi}\|^2}\right)\left(\delta_{in}-\frac{3}{2}\frac{\xi_i\xi_n}{\|\boldsymbol{\xi}\|^2}\right)\frac{\partial^2\delta\psi}{\partial\xi_m\partial\xi_n}+\frac{9}{4}\frac{\|\boldsymbol{\xi}\|^7}{GM^3}\left(\delta_{im}\xi_i+\delta_{mi}\xi_i+\delta_{ii}\xi_m-\frac{7}{2}\frac{\xi_i\xi_m\xi_i}{\|\boldsymbol{\xi}\|^2}\right)\frac{\partial\delta\psi}{\partial\xi_m}+ & \\
+\frac{\|\boldsymbol{\xi}\|^9}{GM^3}\left(\delta_{im}-\frac{3}{4}\frac{\xi_i\xi_m}{\|\boldsymbol{\xi}\|^2}\right)\left(\delta_{in}-\frac{3}{2}\frac{\xi_i\xi_n}{\|\boldsymbol{\xi}\|^2}\right)\frac{\partial^2\delta\psi}{\partial\xi_m\partial\xi_n}-\frac{3}{2}\frac{\|\boldsymbol{\xi}\|^7}{GM^3}\left(\delta_{mi}\xi_i-\frac{1}{2}\delta_{mi}\xi_i+\delta_{ii}\xi_m-\frac{5}{4}\frac{\xi_i\xi_i\xi_m}{\|\boldsymbol{\xi}\|^2}\right)\frac{\partial\delta\psi}{\partial\xi_m} &= 0\,.
\end{aligned} \tag{5.93}$$

It follows

$$\begin{aligned}
-\frac{3}{2}\frac{\|\boldsymbol{\xi}\|^9}{GM^3}\left(\delta_{mn}-\frac{3}{4}\frac{\xi_m\xi_n}{\|\boldsymbol{\xi}\|^2}\right)\frac{\partial^2\delta\psi}{\partial\xi_m\partial\xi_n}+\frac{9}{4}\frac{\|\boldsymbol{\xi}\|^7}{GM^3}\left(\frac{3}{2}\xi_m\right)\frac{\partial\delta\psi}{\partial\xi_m}+ & \\
+\frac{\|\boldsymbol{\xi}\|^9}{GM^3}\left(\delta_{mn}-\frac{9}{8}\frac{\xi_m\xi_n}{\|\boldsymbol{\xi}\|^2}\right)\frac{\partial^2\delta\psi}{\partial\xi_m\partial\xi_n}-\frac{3}{2}\frac{\|\boldsymbol{\xi}\|^7}{GM^3}\left(\frac{9}{4}\xi_m\right)\frac{\partial\delta\psi}{\partial\xi_m} &= 0
\end{aligned} \tag{5.94}$$

from (5.93) by means of simple calculations. Consequently, (5.94) results in

$$-\frac{1}{2}\frac{\|\boldsymbol{\xi}\|^9}{GM^3}\delta_{mn}\frac{\partial^2\delta\psi}{\partial\xi_m\partial\xi_n}=0$$

due to the fact that both terms related to the derivative $\frac{\partial\delta\psi}{\partial\xi_m}$ cancel each other out. At last, since $\|\boldsymbol{\xi}\|\neq 0$ holds in regular gravity space, proof is given that the adjoint disturbing potential $\delta\psi$ fulfills up to terms of higher order Laplace's equation

$$\begin{aligned}
-\frac{1}{2}\frac{\|\boldsymbol{\xi}\|^9}{GM^3}\Delta\delta\psi &= 0 \\
\Delta\delta\psi &= 0\,. \qquad \diamondsuit
\end{aligned}$$

Hence, the subject matter of finding the corresponding field equation for the adjoint disturbing potential $\delta\psi$ in regular gravity space comes to a satisfactory end. It applies that a *homogeneous* partial differential equation, i.e. Laplace's equation, has to be satisfied by $\delta\psi$, which is in contrast to the gravity space approach according to F. Sansò. There the underlying field equation, (4.63), represents an *inhomogeneous* partial differential equation. Naturally, the next question to be approached is related to the associated boundary condition. Starting from the boundary relation (5.69) given in the context of the nonlinear problem, i.e.

$$-\left(\frac{1}{2}\boldsymbol{\xi}^\top\nabla_\xi\psi+\psi\right)\Big|_\Sigma=v\,, \tag{5.95}$$

the following equation

$$-\left(\frac{1}{2}\boldsymbol{\xi}^\top\nabla_\xi\psi_0+\psi_0\right)\Big|_\Sigma=v_0\,. \tag{5.96}$$

is formally deduced by replacing the true adjoint potential ψ by the adjoint normal potential ψ_0. In spite of the strong methodological similarity in deriving (4.58) and (5.96), a major difference between these two relationships exists, which must be pointed out. In contrast to (4.58), where the boundary values v_0 were associated with the boundary surface Σ_g, i.e. the gravimetric telluroid given in geometry space, the situation in (5.96) is somewhat different. Against the background of Lemma 16 implicating the property of identical mapping if the underlying potential functions are of type $\frac{GM}{\|\mathbf{x}\|}$ and $-2\frac{GM}{\|\boldsymbol{\xi}\|}$ accordingly, the boundary values v_0 must refer to the boundary surface Σ as well. That is, the surface Σ, which usually defines the gravimetric telluroid surface in regular gravity space, is mapped in view of Lemma 16 onto the same surface Σ in geometry space and, consequently,

$$v_0=V_0\big|_\Sigma \tag{5.97}$$

holds for the boundary values involved in (5.96).

Now, readopting the modus operandi already familiar from Section 4-5 to determine (4.59), the next step represents simply a subtraction of the two equations above, which is in fact possible since the boundary conditions (5.95) and (5.96) are both linear in terms of ψ and ψ_0. Hence, the result is the boundary condition for the adjoint disturbing potential

$$-\left(\frac{1}{2}\boldsymbol{\xi}^\top\nabla_\xi\delta\psi+\delta\psi\right)\Big|_\Sigma=\Delta v\,, \tag{5.98}$$

where the corresponding boundary values, i.e. gravitational potential anomalies Δv, are obtained as the difference of the measured data v and the values v_0, deduced by evaluation of the normal potential functional at the gravimetric telluroid surface Σ

$$\Delta v = v - v_0 = v - V_0\big|_\Sigma \, . \tag{5.99}$$

Thus, in view of Theorem 8 and (5.98), the resulting BVP can be introduced in the next definition:

Definition 23 *The linear geodetic boundary value problem in regular gravity space comprises the following problem: the data, i.e. gravitational potential anomalies, are now considered to be given at the known surface* Σ

$$\Delta v : \Sigma \to \mathbb{R}$$

and a real function $\delta\psi(\boldsymbol{\xi}) : ext\ \Sigma \to \mathbb{R}$ *is to be found*

$$\Delta\delta\psi(\boldsymbol{\xi}) = 0\,, \quad \boldsymbol{\xi} \in ext\ \Sigma \tag{5.100}$$

$$-\left(\frac{1}{2}\boldsymbol{\xi}^\top \nabla_\xi \delta\psi + \delta\psi\right)\bigg|_\Sigma = \Delta v\,, \tag{5.101}$$

which is the solution of a linear and homogeneous partial differential equation of second order, (5.100), under the boundary condition (5.101).

In contrast to the BVP identified in Definition 19, which is based on Poisson's equation, the underlying field equation in Definition 23 is Laplace's equation. Consequently, the adjoint disturbing potential $\delta\psi$ is only harmonic in regular gravity space and as usual satisfies a linear oblique boundary condition. Again, by means of (5.100) and (5.101), a BVP with a *fixed* boundary is established, since in contrast to the Earth's surface σ, the boundary Σ is a known surface. More specifically, the position vector $\boldsymbol{\xi}\big|_\Sigma$ in regular gravity space describes the known boundary surface Σ as the image of the Earth's surface σ, characterized by the position vector $\mathbf{x}\big|_\sigma$, under the regular gravity space mapping

$$\boldsymbol{\xi}\big|_\Sigma = -\sqrt{GM}\frac{\nabla V(\mathbf{x}\big|_\sigma)}{\|\nabla V(\mathbf{x}\big|_\sigma)\|^{3/2}}\,. \tag{5.102}$$

Now, after having obtained the solution for the adjoint disturbing potential $\delta\psi(\boldsymbol{\xi})$ and by means of (5.82) also for the adjoint potential

$$\psi(\boldsymbol{\xi}) = \psi_0(\boldsymbol{\xi}) + \delta\psi(\boldsymbol{\xi})\,,$$

the Earth's surface σ is derived starting from (5.70) and by taking (5.82) into account

$$\begin{aligned}\mathbf{x}\big|_\sigma = \boldsymbol{\gamma}(\boldsymbol{\xi}\big|_\Sigma)\boldsymbol{\pi}(\boldsymbol{\xi}\big|_\Sigma) = \boldsymbol{\gamma}(\boldsymbol{\xi}\big|_\Sigma)\nabla_\xi\psi(\boldsymbol{\xi}\big|_\Sigma) &= \boldsymbol{\gamma}(\boldsymbol{\xi}\big|_\Sigma)\nabla_\xi\left(\psi_0(\boldsymbol{\xi}\big|_\Sigma) + \delta\psi(\boldsymbol{\xi}\big|_\Sigma)\right)\\ &= \boldsymbol{\gamma}(\boldsymbol{\xi}\big|_\Sigma)\nabla_\xi\psi_0(\boldsymbol{\xi}\big|_\Sigma) + \boldsymbol{\gamma}(\boldsymbol{\xi}\big|_\Sigma)\nabla_\xi\delta\psi(\boldsymbol{\xi}\big|_\Sigma)\,.\end{aligned} \tag{5.103}$$

Then, by insertion of the known expression for $\boldsymbol{\gamma}$, Lemma 21, into (5.103) and by evaluating $\nabla_\xi\psi_0(\boldsymbol{\xi}\big|_\Sigma)$ follows

$$\begin{aligned}\mathbf{x}\big|_\sigma &= \left(-\frac{\|\boldsymbol{\xi}\|^3}{GM}\left[\mathbf{I} - \frac{3}{2}\frac{\boldsymbol{\xi}\boldsymbol{\xi}^\top}{\|\boldsymbol{\xi}\|^2}\right]\frac{2GM}{\|\boldsymbol{\xi}\|^3}\boldsymbol{\xi}\right)\bigg|_\Sigma + \boldsymbol{\gamma}(\boldsymbol{\xi}\big|_\Sigma)\nabla_\xi\delta\psi(\boldsymbol{\xi}\big|_\Sigma)\\ &= \boldsymbol{\xi}\big|_\Sigma + \boldsymbol{\gamma}(\boldsymbol{\xi}\big|_\Sigma)\nabla_\xi\delta\psi(\boldsymbol{\xi}\big|_\Sigma)\end{aligned} \tag{5.104}$$

from straightforward computations. At last, the first term on the right hand side of (5.104), having in mind its dependency on the reference potential ψ_0, accordingly V_0, is handled by taking advantage of the identical mapping property constituted by Lemma 16. Moreover, by introducing the new quantity $\boldsymbol{\zeta}$, cf. also (4.67), for the second term on the right hand side of (5.104), leads to

$$\mathbf{x}\big|_\sigma = \mathbf{x}\big|_\Sigma + \boldsymbol{\zeta}\,, \tag{5.105}$$

i.e. to the position vector $\mathbf{x}\big|_\sigma$ determining the required Earth's surface σ, see also (4.65). In conclusion, the solution $\delta\psi(\boldsymbol{\xi})$ of the linear GBVP in regular gravity space, Definition 23, provides the position correction or position anomaly vector, see Fig. 3.1,

$$\boldsymbol{\zeta} = \boldsymbol{\gamma}(\boldsymbol{\xi}\big|_\Sigma)\nabla_\xi\delta\psi(\boldsymbol{\xi}\big|_\Sigma) \tag{5.106}$$

to determine σ as specified in (5.105), that is, by means of a summation of the vector $\mathbf{x}\big|_\Sigma$, constituting the approximation surface Σ in ordinary space, and the anomaly vector $\boldsymbol{\zeta}$. The specific characteristic of the gravimetric telluroid Σ in gravity space constituted by the vector $\boldsymbol{\xi}\big|_\Sigma$, and its interrelationship to the approximation surface Σ in geometry space founded on the vector $\mathbf{x}\big|_\Sigma$, is more closely discussed among other things in the context of a short summary at the end of this chapter.

5-4 Spherical approximation of the GBVP in regular gravity space

In the style of Section 3-6, where the spherical approximation of the linear Molodensky problem is elaborated, this section aims to introduce an equivalent form of the linear GBVP in regular gravity space. Whereas the derivation of the simple Molodensky's problem in geometry space involves for the approximation of the boundary operator the isotropic normal potential according to (2.40) and the application of spherical coordinates, cf. Section 2-1.1, the argument to simplify the boundary operator in the present context, i.e. (5.101), is of slightly different nature. The reason is that the underlying reference potential ψ_0, introduced in the previous linearization step, already constitutes the auxiliary space counterpart of a spherical normal potential. All the same, the use of a spherical coordinate representation to rewrite the boundary operator (5.101) is intended in the present section.

For that purpose, the position vector $\mathbf{x}$ in ordinary space

$$\mathbf{x} = r \begin{bmatrix} \cos\lambda\cos\phi & \sin\lambda\cos\phi & \sin\phi \end{bmatrix}^\top \tag{5.107}$$

and the associated position vector $\boldsymbol{\xi}$ in regular gravity space

$$\boldsymbol{\xi} = r_\xi \begin{bmatrix} \cos\lambda_\xi\cos\phi_\xi & \sin\lambda_\xi\cos\phi_\xi & \sin\phi_\xi \end{bmatrix}^\top \tag{5.108}$$

are first of all expressed in terms of the polar coordinates (λ, ϕ, r), cf. (2.1).

Based on the familiar relation

$$\boldsymbol{\xi}^\top \nabla_\xi \psi = r_\xi \frac{\partial \psi}{\partial r_\xi}\,, \tag{5.109}$$

it is now permissible to replace the derivative $\nabla_\xi \delta\psi$ in (5.101), simply by the partial derivative with respect to the radial direction only. Consequently, the partial derivatives with respect to λ_ξ and ϕ_ξ, see also (2.62), are neglected in the following. Hence, the BVP in regular gravity space, which corresponds to the simple Molodensky's problem characterized in Definition 11, stands for the problem:

Definition 24 *In regular gravity space, the geodetic boundary value problem in spherical approximation reflects the following problem: the given data, i.e. gravitational potential anomalies, are assigned to the known surface* Σ

$$\Delta v : \Sigma \to \mathbb{R}$$

and to be found is a real function $\delta\psi(\boldsymbol{\xi}) : ext\ \Sigma \to \mathbb{R}$

$$\Delta\delta\psi(\boldsymbol{\xi}) = 0\,, \quad \boldsymbol{\xi} \in ext\ \Sigma \tag{5.110}$$

$$-\left(\frac{1}{2} r_\xi \frac{\partial \delta\psi}{\partial r_\xi} + \delta\psi\right)\Bigg|_\Sigma = \Delta v\,, \tag{5.111}$$

which is the solution of a linear and homogeneous partial differential equation of second order, (5.110), under the boundary condition (5.111).

The appropriateness of the above BVP is supported by the following reasoning. Starting from Remark 2, emphasizing the prevailing radial dependency of the potential V, and by means of Section 5-1.1, which constitutes a comparable asymptotic behaviour of V and ψ, it can be concluded that the adjoint potential ψ is also dominated by a radial dependency, see also Fig. 24. Now the assumption is that to a certain extent, at least on a global average, this radial predominance also holds for the disturbing potential functions, such as the disturbing gravitational potential δV and the according disturbing adjoint potential $\delta\psi$. Then, under the circumstances that $\delta\psi$ is a small and radially dependent quantity, the BVP according to Definition 24 appears feasible.

Furthermore, the BVP according to Definition 24 is of the same mathematical structure as the BVP introduced previously in Definition 23. That is, (5.110) and (5.111) constitute a linear fixed BVP, which is based on a Laplace type of differential equation and is subject to an oblique boundary condition. Again, the known surface Σ is obtained from (5.102) and the Earth's surface σ results from (5.105) and (5.106) after having solved the spherically approximated problem defined above for $\delta\psi(\boldsymbol{\xi})$.

Triggered principally by practical considerations, another modification, i.e. constant radius approximation with respect to the boundary condition (5.111), will be pursued in the next section.

5-5 Constant radius approximation of the GBVP in regular gravity space

Providing a version of the GBVP in regular gravity space, which will form the theoretical basis for the practical solution of the problem, is the intention of this section. This will be conducted in a manner similarly to Section 3-7. For that purpose, the longitudinal and latitudinal discrepancies of (5.107) and (5.108) are disregarded in a first approximation. Thus, the following identities hold

$$\lambda_\xi = \lambda \quad ; \quad \phi_\xi = \phi \,. \tag{5.112}$$

This modus operandi is motivated against the background of the identical mapping characteristic proposed in Lemma (16) and displayed in Fig 5.2. In this regard the congruency of $\boldsymbol{\xi}$ and $\mathbf{x}$ has been pointed out, which was found to also apply approximately, i.e.

$$\boldsymbol{\xi} \approx \mathbf{x} \,, \tag{5.113}$$

if the restriction $V = V_0$ is given up. Consequently, in view of (5.113), (5.112) can be explained. Moreover, the radial coordinate r_ξ in (5.108) is split up into the constant R, which denotes the radius of the underlying reference sphere S, and the height h_ξ above S

$$r_\xi = (R + h_\xi) \;. \tag{5.114}$$

Then, by taking (5.112) and (5.114) into account, (5.108) becomes

$$\boldsymbol{\xi} = (R + h_\xi) \begin{bmatrix} \cos\lambda\cos\phi & \sin\lambda\cos\phi & \sin\phi \end{bmatrix}^\top \,. \tag{5.115}$$

By means of this step, associated points on the actual boundary surface Σ and on the reference sphere S are located in the same radial direction. As will be discussed later, this is of importance with regard to the subsequent numerical evaluation of the GBVP in regular gravity space.
The second approximation step is motivated by the same argument as given before in (3.71) in conjunction with the investigations of the simple Molodensky's problem in ordinary space. In detail, the assumption is again that the difference of r_ξ and R is negligible and that an error of h_ξ/R is tolerable. Thus, it applies that

$$h_\xi = 0$$

holds and, as a result, (5.115) reduces to

$$\boldsymbol{\xi}\big|_S = R \begin{bmatrix} \cos\lambda\cos\phi & \sin\lambda\cos\phi & \sin\phi \end{bmatrix}^\top \,. \tag{5.116}$$

Hence, the BVP in regular gravity space presented in the following originates from a simple radial mapping of the boundary data from the boundary surface Σ to the corresponding reference sphere S given by $\boldsymbol{\xi}\big|_S$:

Definition 25 *In regular gravity space, the geodetic boundary value problem in constant radius approximation treats the following problem: the given data, i.e. gravitational potential anomalies, are now assigned to the reference sphere S*

$$\Delta v : S \to \mathbb{R}$$

and to be found is a real function $\delta\psi(\boldsymbol{\xi}) : ext\ S \to \mathbb{R}$

$$\Delta\delta\psi(\boldsymbol{\xi}) = 0 \,, \quad \boldsymbol{\xi} \in ext\ S \tag{5.117}$$

$$-\left(\frac{1}{2} r_\xi \frac{\partial\delta\psi}{\partial r_\xi} + \delta\psi\right)\bigg|_S = \Delta v \,, \tag{5.118}$$

which is the solution of a linear and homogeneous partial differential equation of second order, (5.117), under the boundary condition (5.118).

The above representation of the GBVP in regular gravity space is particularly suited for computational purposes since the underlying boundary surface reduces to the sphere S as stated in (5.118). Indeed, the sphere S represents a very simple surface predestined to perform a spherical harmonic analysis, as will be accomplished in Chapter 7, where within the scope of a *proof of concept* the numerical solution of the problem specified by Definition 25 is achieved.

Furthermore, the BVP introduced above constitutes a linear fixed problem based on a *normal derivative* boundary condition. That is, the direction of the derivative $\frac{\partial}{\partial r_\xi}$ is orthogonal to the surface of the sphere S. This is in contrast to the problem given in Definition 24, where $\frac{\partial}{\partial r_\xi}$ represents an oblique derivative with respect to the surface Σ. As turns out in Chapter 7, a normal derivative problem proves to be advantageous for the computational problem solution compared to an oblique derivative problem.

At last, the problem of determining the Earth's surface σ has to be addressed again. Despite the utilization of S in (5.118), the surface Σ, when considered in ordinary space, still establishes the approximation surface for σ. The reason is that S, similarly to Section 3-7, again serves only as a computational reference surface in order to solve for $\delta\psi(\boldsymbol{\xi})$. Thus, with Σ obtained from (5.102), the Earth's surface σ results from (5.105) and (5.106), thereby using $\delta\psi(\boldsymbol{\xi})$ as provided before.

Remark 26 As previously discussed in conjunction with the examination of Molodensky's problem in ordinary space, see also Remark 10, it is advisable, in order to meet the ever increasing requirements in terms of accuracy, to project the boundary data not only onto the sphere, but to use numerical techniques, e.g. collocation, in order to perform explicit continuation of the boundary data. As a matter of fact, data continuation by means of collocation further provides the opportunity to give up the necessity of radial projection, (5.112). This may be necessary if the actual data on the gravimetric telluroid is distributed differently from the structure of the data grid required for the computational procedures such as the spherical harmonic analysis on the sphere. Moreover, as outlined already in Remark 10 it might be worthwhile to consider the use of an elliptically-shaped computational surface, since such a surface better reflects the actual geometry, thus reducing the separation distance of the actual boundary surface and the mathematical reference surface, which is important with regard to the proposed data continuation.
Besides such geometrical considerations with respect to processing the boundary data, it is worthwhile to give some thought to a possible improvement of the applied physics within the approximation step. That is, a more sophisticated normal potential than the one applied so far based on a spherical configuration can be imagined. In fact, the possibility of adopting an ellipsoidal normal potential is investigated in the next chapter right after the following short summary.

5-6 Brief summary on the status quo of the BVPs presented so far

As outlined before in Remark 25, the corresponding nonlinear BVPs of Section 4-5 and Section 5-3 were found to be of a similar mathematical structure. Only a fine difference between the gravity space and the regular gravity space concept in the nonlinear case has been pointed out. Both problems differ in the elements of the matrices $\boldsymbol{\Psi}$ and $\boldsymbol{\Phi}$, respectively. However, the situation revealed in Section 5-3 with regard to the linear problem turns out to be of a different nature. In fact, two fundamental differences between the linear GBVP in gravity space and the linear GBVP in regular gravity space have been elaborated throughout this chapter. In the first place, the field equation within the context of F. Sansò's gravity space approach has been identified as an inhomogeneous partial differential equation of Poisson type, which is in contrast to the homogeneous partial differential equation of Laplace type associated with the regular gravity space approach. It is obvious that the latter problem based on Laplace's equation is particularly favorable in terms of complexity and computational solvability. Moreover, as far as the boundary values v_0 related to the spherical reference potential $V_0 = \frac{GM}{\|\mathbf{x}\|}$ are concerned, the following statements can be made. Basically, F. Sansò's linear gravity space concept is based on *three* distinctive surfaces, cf. Section 4-5. First of all the Earth's surface σ, secondly the corresponding image of σ in gravity space, i.e. the boundary surface Σ and thirdly the surface Σ_g, which establishes the reference surface for the normal potential values v_0 required to eventually derive the potential anomalies Δv. Now, as opposed to the simple gravity space concept, the regular gravity space approach features only two relevant reference surfaces. Namely, only the terrestrial surface σ and its regular gravity space image Σ are involved. Admittedly, it should be acknowledged that the boundary values v_0 in (5.96) refer to an independent surface, which differs from Σ in the strictest sense. The reason is that the boundary surface associated with the normal data v_0 lies in ordinary space, whereas Σ constitutes a surface in regular gravity space. However, due to the identical mapping behaviour postulated by Lemma 16, which applies in this special case, the surface Σ and its image in ordinary space are regarded as the same surface. The existence of only two instead of three reference surfaces within the scope of the regular gravity space theory is advantageous for two reasons. On the one hand this speciality will be significant with regard to the implementation of the numerical BVP solution, and on the other hand it simplifies matters from the theoretical point of view.

Next, in order to continue the comparison of gravity space and regular gravity space features, it can be noted that in the framework of both approaches, the nonlinear as well as the linear BVPs feature linear oblique boundary conditions. Moreover, due to the aforementioned anharmonicity of the adjoint disturbing potential, the investigation of the GBVP in gravity space has been abandoned after derivation of the linear problem whereas the treatment of the GBVP in regular gravity space has been carried on. The modifications according to Section 5-4 and Section 5-5, i.e. spherical and constant radius approximation, yield a further simplified representation for the GBVP in regular gravity space. However, these variants of the GBVP are required for the actual numerical experiment. This will be accomplished in Chapter 7, where within a proof of concept other questions are also clarified. E.g., about the permissibility of these approximations and, if necessary, how resulting difficulties could be addressed.

Besides a comparison of the underlying BVPs in gravity space and in regular gravity space, another matter of particular interest is the strong similarity between the simple Molodensky's problem outlined in Definition 11 and the version of the GBVP in regular gravity space according to the previous Definition 25. In fact, both BVPs are of equivalent mathematical structure. They differ only by the status, which the potential functions V, W and their first order functionals $\mathbf{g}$, $\mathbf{\Gamma}$ have. That is, in the framework of Molodensky's theory, the boundary surface relies on gravity potential data and the boundary information is constituted by gravity anomalies. This is in contrast to the methodology set forth in this chapter, where the definition of the boundary surface requires knowledge of the gradient vectors of the gravitational potential, whereas the underlying boundary values are gravitational potential anomalies. The benefit of using potential anomalies instead of gravity anomalies as boundary data is also validated in Chapter 7. Furthermore, it should be highlighted that due to the mathematical equivalence of the classical approach to the GBVP and the approach based on regular gravity space, it is of course possible to still apply all known ideas and methods developed for solving Molodensky's problem.

At last, similarly to the gravity space approach according to F. Sansò, Chapter 4, the new regular approach presented in the current chapter also implies a spherical configuration. That is, within the linearization process of the BVP, application of a spherical normal potential is involved in order to approximate the actual potential, cf. Sections 4-5 and 5-3. Numerical experiments, as will be shown later in Chapter 7, indicate that a spherical configuration, which represents an approximation of first order only, implicates computational difficulties. However, adopting a better approximate for the true potential function raises expectations that these difficulties can be abolished, see also [88] Seitz 1997. Therefore, a second and, to a certain extent, new approach is based on an ellipsoidal configuration. Consequently, an ellipsoidal normal potential is utilized in the framework of linearization. On this account, in the next chapter, the line of argument given in the work of W. Keller is recapitulated taking the modified linearization point into consideration.

Chapter 6

A boundary value approach in ellipsoidal regular gravity space

The two variants of treating the GBVP in gravity space, namely F. Sansò's gravity space approach and, as seen in the preceding chapter, the regular gravity space approach according to W. Keller, both have a spherical linearization point in common. This chapter will shed light on the question of whether a linearization concept other than the one based on a spherical scenario might be favorable. Hence, the investigations outlined in this chapter imply an *ellipsoidal* configuration type of regular gravity space theory and are arranged as follows. In fact, a more precise notation would be *spheroidal* instead of ellipsoidal configuration since a truncated series expansion representation in J_2 for the normal potential will be applied. However, for the sake of simplicity and understandability the term ellipsoidal will be used henceforth. In the next section, this new regular gravity space approach is derived and its associated properties are elaborated. Section 6-2 is dedicated to the problem of finding the nonlinear representation of the GBVP for ellipsoidal regular gravity space. As a matter of fact, the resulting nonlinear BVP exhibits a very high degree of similarity to the nonlinear BVP given in case of the spherical regular gravity space theory as developed in the last chapter. Section 6-3 gets into the question of the related linearization and the corresponding linear problem is deduced. Within the scope of the linearization process, it turns out that the ellipsoidal approach can indeed be regarded as a perturbation of the thoroughly investigated spherical variant. Thus, under certain circumstances it is possible to reduce the ellipsoidal formulation to the spherical formulation. The subject matter of deriving the corresponding spherical approximation step is addressed in Section 6-4 and consequently the resulting linear BVP in spherical approximation is obtained. As before, the field equation, which the adjoint disturbing potential has to fulfill, is a partial differential equation of Laplace type subject to a linear boundary condition. Thereafter, a brief discussion on the benefit of the ellipsoidal methodology concludes this chapter.

6-1 An ellipsoidal regular gravity space transformation

In order to establish an ellipsoidal formulation, an alternative reasoning centered on the identical mapping property, cf. Lemma 16, to derive the regular gravity space transformation (5.27) is first pointed out. The reason is that by following this example a nearly analogous procedure, simply replacing the spherical configuration by an ellipsoidal configuration, is adopted in order to introduce the new regular gravity space formulation. However, as far as the new *ellipsoidal regular gravity space transformation* is concerned, it has to be acknowledged that no closed solution for the new formulae can be obtained. In fact, an implicit definition has to be introduced as a start. Accordingly, a practical representation of the transformation formula is achieved by means of a series expansion with respect to a small parameter describing the deviation from the spherical scenario. Section 6-1.1 addresses this subject, thereby establishing the required relationship between a coordinate vector related to ordinary space and a coordinate vector related to the new auxiliary space. Hence, in view of the newly installed transformation, *ellipsoidal regular gravity space* and the associated boundary surface Σ are formally introduced in Section 6-1.2. Furthermore, the desired identical mapping property of the ellipsoidal regular gravity space transformation in case of an ellipsoidal reference potential is confirmed in this context. Thereafter, in Section 6-1.3, a contact transformation type of interpretation will be the prevailing form of looking at the new ellipsoidal regular gravity space transformation, which, in that case, is exclusively regarded as a relationship of the new coordinate vector and the old momentum vector. Together with two other relations that will be derived for the transformation between ordinary potential and adjoint potential functional as well as for the determination of the new momentum quantities, the *ellipsoidal regular gravity space approach* is completed. It will turn out that within the framework of the ellipsoidal regular gravity space approach, the formulae for determining the adjoint potential and the new momenta are of the same mathematical structure as given in the last chapter for the regular gravity space approach according to W. Keller. Thus, the only difference is the advanced coordinate mapping between geometry and gravity space in the case under consideration.

As indicated before, W. Keller's regular gravity space transformation, cf. (5.27), can be derived by following an alternative line of argument, thereby starting from the identical mapping property in case of a spherical linearization point in order to deduce the corresponding transformation formulae. This approach will also serve as the basis to develop the proposed ellipsoidal transformation later on. Thus, based on the already familiar definition

$$\mathbf{p} := \nabla V(\mathbf{x})$$

of the vector $\mathbf{p}$ as the gradient vector of the gravitational potential V, see e.g. Remark 16, the new coordinate vector $\boldsymbol{\xi}$ is identified in the first instance simply as a function of $\mathbf{p}$

$$\boldsymbol{\xi} = \boldsymbol{\xi}(\mathbf{p})\,. \tag{6.1}$$

Now, the idea is that the function $\boldsymbol{\xi}(\mathbf{p})$ can be uniquely defined by setting certain constraints that have to be satisfied. Based on these constraints, it is possible to derive the corresponding analytical expression for $\boldsymbol{\xi}(\mathbf{p})$. At best an explicit form for $\boldsymbol{\xi}(\mathbf{p})$ can be found, which is of course the primary aim.

The first assumption is that by adopting the isotropic potential V_0, cf. (2.40), as an approximation for the true potential V, the vector $\mathbf{p}$ is specified according to

$$\tilde{\mathbf{p}} = \nabla V_0 = \nabla\left(\frac{GM}{\|\mathbf{x}\|}\right) = -\frac{GM}{\|\mathbf{x}\|^3}\mathbf{x}\,. \tag{6.2}$$

Moreover, the second requirement is that the following identity applies

$$\boldsymbol{\xi}(\tilde{\mathbf{p}}) = \mathbf{x}\,. \tag{6.3}$$

Consequently, it holds that by means of the last two relationships the transformation (6.1) is properly defined and that it is possible to derive (6.1) explicitly from (6.2) and (6.3). This assertion can be verified by making use of the identity $\boldsymbol{\xi} = \mathbf{x}$ in (6.2)

$$\tilde{\mathbf{p}}(\boldsymbol{\xi}) = -\frac{GM}{\|\boldsymbol{\xi}\|^3}\boldsymbol{\xi} \tag{6.4}$$

and by solving the resulting expression (6.4) for $\boldsymbol{\xi}$, that is

$$\boldsymbol{\xi}(\tilde{\mathbf{p}}) = -\sqrt{GM}\frac{\tilde{\mathbf{p}}}{\|\tilde{\mathbf{p}}\|^{3/2}}\,. \tag{6.5}$$

In general, the requested transformation

$$\boldsymbol{\xi}(\mathbf{p}) = -\sqrt{GM}\frac{\mathbf{p}}{\|\mathbf{p}\|^{3/2}} \tag{6.6}$$

is simply obtained by giving up the restriction according to (6.2).

Now, these relations are reconsidered in the further course, while instead of the isotropic normal potential the following ellipsoidal or more precisely, as discussed in the beginning of this chapter, spheroidal reference potential

$$V_0^{ell}(\mathbf{x}) = \frac{GM}{r}\left(1 - J_2\left(\frac{R}{r}\right)^2 P_2(\sin\phi)\right) \tag{6.7}$$

$$= \frac{GM}{\|\mathbf{x}\|}\left(1 - J_2\left(\frac{R}{\|\mathbf{x}\|}\right)^2\left(\frac{3}{2}\left(\frac{x_3}{\|\mathbf{x}\|}\right)^2 - \frac{1}{2}\right)\right) \tag{6.8}$$

is employed, cf. [28] HEISKANEN&MORITZ 1967; [7] BRETTERBAUER 1988 and [67] MORITZ 1990. Eq. (6.7) defines the new gravitational normal potential $V_0^{ell}(\mathbf{x})$ in terms of a rotationally symmetric zonal expansion. It results practically from (2.54), thereby limiting the expansion simply to the central term $(0,0)$ and the flattening

term $(2,0)$. Moreover, in contrast to (2.54), which is based on the use of fully normalized associated Legendre functions $P^*_{kl}(\sin\phi)$, see also Section 2-2.4 and Appendix A, (6.7) is expressed in terms of the Legendre polynomial $P_2(\sin\phi) = \frac{3}{2}\sin^2\phi - \frac{1}{2}$. Consequently, instead of the normalized Stokes coefficient c^*_{20}, the small parameter J_2 is applied, which is basically related to the former parameter according to $J_2 = -\sqrt{5}c^*_{20}$. At last, the Cartesian representation (6.8) is deduced from (6.7) by taking (2.4) and (2.7) into account.
Adopting the potential V_0^{ell} according to (6.8) as an improved approximation of the true potential is motivated by the fact that the gravimetric telluroid Σ is the image of the Earth's surface σ under the gravity space mapping (5.1) or rather (6.6). More precisely, it is a specific characteristic, as recalled above, that the gravity space mapping (6.6) yields the identical transformation for the isotropic potential V_0. Hence, due to the fact that the actual potential V differs from the isotropic potential V_0 by about 10^{-3}, also the gravimetric telluroid Σ, when utilized as a reference surface in geometry space as discussed in Section 5-6, differs from the Earth's surface σ by about the same factor. This leads to a separation between Σ and σ of several kilometers, which also involves an increase in magnitude of the underlying boundary data as can be directly deduced from (5.99). From a computational point of view, this situation should preferably be avoided. One way to reduce the separation between Σ and σ is to find a new gravity space transformation, which is the identical transformation for the more sophisticated spheroidal reference potential V_0^{ell}. This shall be attempted next.

In principle, it has been the intention to revise the modus operandi according to (6.1)-(6.6) for the modified reference potential. That is, instead of (6.2), it holds

$$\tilde{\mathbf{p}} = \nabla V_0^{ell}(\mathbf{x})\,. \tag{6.9}$$

Again, it is required that the following identity

$$\boldsymbol{\xi}(\tilde{\mathbf{p}}) = \mathbf{x} \tag{6.10}$$

is true, yet in the present case with $\tilde{\mathbf{p}}$ given according to (6.9). Though this time it turns out that in following the example, given in conjunction with (6.4), application of $\boldsymbol{\xi} = \mathbf{x}$ in (6.9), i.e.

$$\tilde{\mathbf{p}} = \nabla_\xi V_0^{ell}(\boldsymbol{\xi})\,, \tag{6.11}$$

is not a successful measure, since the resulting equation cannot be solved directly for $\boldsymbol{\xi}$ due to the advanced complexity of V_0^{ell} in contrast to V_0. Consequently, it is not possible to establish an explicit form for the transformation $\boldsymbol{\xi} = \boldsymbol{\xi}(\mathbf{p}, J_2)$. However, (6.11) can be rewritten in the following way

$$\tilde{\mathbf{p}} - \nabla_\xi V_0^{ell}(\boldsymbol{\xi}) = 0\,, \tag{6.12}$$

which by making use of the implicit function theorem can be solved for $\boldsymbol{\xi} = \boldsymbol{\xi}(\mathbf{p}, J_2)$. For that purpose, the corresponding gravity space transformation is initially specified in an implicit form:

Definition 26 *Let the vector $\boldsymbol{\eta}$ be given by*

$$\boldsymbol{\eta} = \begin{bmatrix} p_1 & p_2 & p_3 & J_2 \end{bmatrix}^\top. \tag{6.13}$$

Furthermore, the function $\boldsymbol{\varrho}$ reads as follows

$$\boldsymbol{\varrho}(\boldsymbol{\eta},\boldsymbol{\xi}) = \mathbf{p} - \nabla_\xi \left(\frac{GM}{\|\boldsymbol{\xi}\|} \left(1 - J_2 \left(\frac{R}{\|\boldsymbol{\xi}\|} \right)^2 \left(\frac{3}{2} \left(\frac{\xi_3}{\|\boldsymbol{\xi}\|} \right)^2 - \frac{1}{2} \right) \right) \right)\,. \tag{6.14}$$

Then, the transformation

$$\boldsymbol{\xi} = \boldsymbol{\xi}(\boldsymbol{\eta}) \tag{6.15}$$

is implicitly defined by

$$\boldsymbol{\varrho}(\boldsymbol{\eta},\boldsymbol{\xi}) = 0\,. \tag{6.16}$$

At first, with regard to the above definition, it is worth mentioning that $\boldsymbol{\eta}$ represents a vector of four independent variables. In addition to the usual three independent variables p_i, cf. Section 5-1.3, the parameter J_2 has to be included within the scope of the proposed extension from a spherical to an ellipsoidal scenario in terms of the reference potential. As before, the vector $\boldsymbol{\xi}$ comprises three dependent variables $\xi_i(\boldsymbol{\eta})$.

Next, it must be verified in the context of a first lemma, that the implicit form (6.16) can indeed be solved for the function $\boldsymbol{\xi}(\boldsymbol{\eta})$:

Lemma 31 *By means of the implicit function theorem existence and uniqueness of the solution*

$$\boldsymbol{\xi} = \boldsymbol{\xi}(\boldsymbol{\eta}) \tag{6.17}$$

from

$$\boldsymbol{\varrho}(\boldsymbol{\eta}, \boldsymbol{\xi}) = 0 \tag{6.18}$$

is guaranteed for J_2 *being small enough.*

Proof. First of all, since it is not always possible to find the function $\boldsymbol{\xi}(\boldsymbol{\eta})$, it is necessary to fix a point $(\boldsymbol{\eta}_0, \boldsymbol{\xi}_0)$ and to consider the domain near that point. Hence, according to the implicit function theorem, e.g. [91] Spivak 1965, it has to be shown that a point $(\boldsymbol{\eta}_0, \boldsymbol{\xi}_0)$ exists, which satisfies

$$\boldsymbol{\varrho}(\boldsymbol{\eta}_0, \boldsymbol{\xi}_0) = 0\,. \tag{6.19}$$

In order to fix a point, it is suitable to choose

$$J_2 = 0\,, \tag{6.20}$$

which basically implies a return to the regular gravity space approach elaborated in the last chapter. Thus, it holds

$$\boldsymbol{\eta}_0 = \begin{bmatrix} p_1 & p_2 & p_3 & 0 \end{bmatrix}^\top \tag{6.21}$$

$$\boldsymbol{\xi}_0 = = -\sqrt{GM}\frac{\mathbf{p}}{\|\mathbf{p}\|^{3/2}}\,, \tag{6.22}$$

and in compliance with the implicit function theorem the corresponding first requirement, (6.19), is satisfied as follows

$$\begin{aligned}
\boldsymbol{\varrho}(\boldsymbol{\eta}_0, \boldsymbol{\xi}_0) &= \mathbf{p} - \nabla_\xi\left(\frac{GM}{\|\boldsymbol{\xi}_0\|}\right) \\
&= \mathbf{p} + \frac{GM}{\|\boldsymbol{\xi}_0\|^3}\boldsymbol{\xi}_0 \\
&= \mathbf{p} + \frac{GM}{\|\sqrt{GM}\frac{\mathbf{p}}{\|\mathbf{p}\|^{3/2}}\|^3}\left(-\sqrt{GM}\frac{\mathbf{p}}{\|\mathbf{p}\|^{3/2}}\right) \\
&= \mathbf{p} - \mathbf{p} = 0\,.
\end{aligned}$$

Next, according to the implicit function theorem it applies furthermore that the function $\boldsymbol{\xi}(\boldsymbol{\eta})$ exists in the near vicinity of $(\boldsymbol{\eta}_0, \boldsymbol{\xi}_0)$, if the following Jacobian matrix

$$\left.\left[\frac{\partial \boldsymbol{\varrho}}{\partial \boldsymbol{\xi}}\right]\right|_{\boldsymbol{\eta}_0, \boldsymbol{\xi}_0} \tag{6.23}$$

is invertible. This immediately leads to the second condition

$$\left.\left|\left[\frac{\partial \boldsymbol{\varrho}}{\partial \boldsymbol{\xi}}\right]\right|\right|_{\boldsymbol{\eta}_0, \boldsymbol{\xi}_0} \neq 0\,, \tag{6.24}$$

that is, the Jacobian, i.e. the determinant of the matrix (6.23), has to be unequal to zero. To give proof that (6.24) is indeed fulfilled, the Jacobian has to be evaluated in detail

$$\begin{aligned}
\left.\left|\left[\frac{\partial \boldsymbol{\varrho}}{\partial \boldsymbol{\xi}}\right]\right|\right|_{\boldsymbol{\eta}_0, \boldsymbol{\xi}_0} &= \left.\left|\left[\frac{\partial}{\partial \boldsymbol{\xi}}\left(\frac{GM}{\|\boldsymbol{\xi}\|^3}\boldsymbol{\xi}\right)\right]\right|\right|_{\boldsymbol{\xi}_0} \\
&= \left|\frac{GM}{\|\boldsymbol{\xi}_0\|^3}\left[\mathbf{I} - 3\frac{\boldsymbol{\xi}_0\boldsymbol{\xi}_0^\top}{\|\boldsymbol{\xi}_0\|^2}\right]\right| \neq 0\,. \qquad \diamondsuit
\end{aligned} \tag{6.25}$$

In order to support the conclusion given in (6.25), it is useful to also consider Lemma 20 and the matrix $\boldsymbol{\alpha}$ constituted in that context. As a matter of fact, the corresponding functional matrices, cf. (5.30) and (6.25), are equivalent. Thus, as much as $\boldsymbol{\alpha}$ has been identified as a regular matrix, since $\boldsymbol{\xi} = 0$ could be excluded from the solution domain, the same argument applies for the Jacobian matrix involved in (6.25) above. Consequently, the relevant determinant is unequal to zero as required.

Once more it has to be emphasized, that the above derivations refer to a distinct fixed point $(\boldsymbol{\eta}_0, \boldsymbol{\xi}_0)$ in which the two important conditions (6.19) and (6.24) are shown to be satisfied. Hence, the solution $\boldsymbol{\xi}(\boldsymbol{\eta})$ exists and is unique only in the near vicinity of this point specified by $J_2 = 0$. This means that the parameter J_2 has to be at the very least a small number to still guarantee existence and uniqueness of $\boldsymbol{\xi}(\boldsymbol{\eta})$ by means of the implicit function theorem. The task of $\boldsymbol{\xi}(\boldsymbol{\eta})$ determination will be addressed in the next section and the question of what has to be understood de facto by *small* in the present context is discussed in the next chapter dealing with the numerical proof of concept.

6-1.1 Series expansion representation of transformation

Having shown in the last section that the implicit form (6.16) can theoretically be solved for $\boldsymbol{\xi}(\boldsymbol{\eta})$, the analytical determination of $\boldsymbol{\xi}(\boldsymbol{\eta})$ is addressed next. However, whereas the implicit function theorem actually only guarantees that such a solution exists and is unique, it is not necessary that an explicit form for $\boldsymbol{\xi}(\boldsymbol{\eta})$ can be provided. In fact, since no closed solution $\boldsymbol{\xi}(\boldsymbol{\eta})$ has been found from the implicit form, the representation of $\boldsymbol{\xi}(\boldsymbol{\eta})$ is intended to be given in terms of a series expansion with respect to the small quantity J_2. On this account, the implicit form (6.16), which is to be solved for $\boldsymbol{\xi}(\boldsymbol{\eta})$ by means of a series expansion approach, is recalled and written as follows

$$\mathbf{p}(\boldsymbol{\xi}, J_2) = \nabla_\xi \left(\frac{GM}{\|\boldsymbol{\xi}\|} \left(1 - J_2 \left(\frac{R}{\|\boldsymbol{\xi}\|} \right)^2 \left(\frac{3}{2} \left(\frac{\xi_3}{\|\boldsymbol{\xi}\|} \right)^2 - \frac{1}{2} \right) \right) \right) . \tag{6.26}$$

The answer to the problem of finding the associated solution is addressed by the following lemma:

Lemma 32 *The series expansion solution of (6.26) with respect to $\boldsymbol{\xi}(\mathbf{p}, J_2)$ is given by*

$$\boldsymbol{\xi}(\mathbf{p}, J_2) = \boldsymbol{\xi}_0(\mathbf{p}) + J_2 \boldsymbol{\xi}_1(\mathbf{p}) + J_2^2 \boldsymbol{\xi}_2(\mathbf{p}) + \mathrm{O}\big(J_2^3\big) \tag{6.27}$$

subject to

$$\begin{aligned}
\boldsymbol{\xi}_0 &= -\sqrt{GM} \frac{\mathbf{p}}{\|\mathbf{p}\|^{3/2}} \\
\boldsymbol{\xi}_1 &= \frac{-3R^2}{4\sqrt{GM}\|\mathbf{p}\|^{5/2}} \begin{bmatrix} p_1(p_1^2 + p_2^2 + 2p_3^2) \\ p_2(p_1^2 + p_2^2 + 2p_3^2) \\ -p_3(3p_1^2 + 3p_2^2 + 2p_3^2) \end{bmatrix} \\
\boldsymbol{\xi}_2 &= \frac{9R^4}{32GM^{3/2}\|\mathbf{p}\|^{7/2}} \begin{bmatrix} p_1(5p_1^4 + 5p_2^4 - 4p_2^2p_3^2 - 60p_3^4 + 2p_1^2(5p_2^2 - 2p_3^2)) \\ p_2(5p_1^4 + 5p_2^4 - 4p_2^2p_3^2 - 60p_3^4 + 2p_1^2(5p_2^2 - 2p_3^2)) \\ p_3(-51p_1^4 - 51p_2^4 + 20p_2^2p_3^2 + 20p_3^4 + p_1^2(-102p_2^2 + 20p_3^2)) \end{bmatrix}
\end{aligned} \tag{6.28}$$

and

$$\mathbf{p} = \begin{bmatrix} p_1 & p_2 & p_3 \end{bmatrix}^\top = \nabla V(\mathbf{x}) . \tag{6.29}$$

Proof. On identifying

$$\mathbf{F}_0(\boldsymbol{\xi}) = \nabla_\xi \left(\frac{GM}{\|\boldsymbol{\xi}\|} \right) \tag{6.30}$$

and

$$\mathbf{F}_1(\boldsymbol{\xi}) = -\nabla_\xi \left(\frac{GM}{\|\boldsymbol{\xi}\|} \left(\frac{R}{\|\boldsymbol{\xi}\|} \right)^2 \left(\frac{3}{2} \left(\frac{\xi_3}{\|\boldsymbol{\xi}\|} \right)^2 - \frac{1}{2} \right) \right) \tag{6.31}$$

in (6.26), the equation to be solved for $\boldsymbol{\xi}(\mathbf{p}, J_2)$ is of type

$$\mathbf{p}(\boldsymbol{\xi}, J_2) = \mathbf{F}_0(\boldsymbol{\xi}) + J_2 \mathbf{F}_1(\boldsymbol{\xi}) . \tag{6.32}$$

For $J_2 = 0$ the solution of the resulting equation

$$\mathbf{p}(\boldsymbol{\xi}) = \mathbf{F}_0(\boldsymbol{\xi}_0)$$

is obviously

$$\boldsymbol{\xi}_0(\mathbf{p}) = -\sqrt{GM} \frac{\mathbf{p}}{\|\mathbf{p}\|^{3/2}} , \tag{6.33}$$

which is necessarily in agreement with the previous regular gravity space approach based on the spherical configuration. However, as discussed in conjunction with Lemma 31, the implicit function theorem now guarantees that for a small enough J_2, (6.32) also has a unique solution $\boldsymbol{\xi}(\mathbf{p}, J_2)$, which is analytical with respect to J_2

$$\boldsymbol{\xi}(\mathbf{p}, J_2) = \boldsymbol{\xi}_0(\mathbf{p}) + J_2\boldsymbol{\xi}_1(\mathbf{p}) + J_2^2\boldsymbol{\xi}_2(\mathbf{p}) + \mathrm{O}\big(J_2^3\big) \ . \tag{6.34}$$

Hence, inserting (6.34) into (6.32) and expanding it into a series with respect to J_2, omitting terms of order $\mathrm{O}\big(J_2^3\big)$ and higher, yields

$$\begin{aligned}
\mathbf{p} &= \mathbf{F}_0(\boldsymbol{\xi}_0 + J_2\boldsymbol{\xi}_1 + J_2^2\boldsymbol{\xi}_2) + J_2\mathbf{F}_1(\boldsymbol{\xi}_0 + J_2\boldsymbol{\xi}_1 + J_2^2\boldsymbol{\xi}_2) \\
&= \mathbf{F}_0(\boldsymbol{\xi}_0) + \nabla\mathbf{F}_0(\boldsymbol{\xi}_0)(J_2\boldsymbol{\xi}_1 + J_2^2\boldsymbol{\xi}_2) + \frac{1}{2}(J_2\boldsymbol{\xi}_1 + J_2^2\boldsymbol{\xi}_2)^\top\nabla^2\mathbf{F}_0(\boldsymbol{\xi}_0)(J_2\boldsymbol{\xi}_1 + J_2^2\boldsymbol{\xi}_2) + \\
&\quad + J_2\mathbf{F}_1(\boldsymbol{\xi}_0) + \nabla\mathbf{F}_1(\boldsymbol{\xi}_0)(J_2^2\boldsymbol{\xi}_1) \\
&= \mathbf{F}_0(\boldsymbol{\xi}_0) + J_2\Big(\nabla\mathbf{F}_0(\boldsymbol{\xi}_0)\boldsymbol{\xi}_1 + \mathbf{F}_1(\boldsymbol{\xi}_0)\Big) + J_2^2\left(\nabla\mathbf{F}_0(\boldsymbol{\xi}_0)\boldsymbol{\xi}_2 + \frac{1}{2}\boldsymbol{\xi}_1\nabla^2\mathbf{F}_0(\boldsymbol{\xi}_0)\boldsymbol{\xi}_1 + \nabla\mathbf{F}_1(\boldsymbol{\xi}_0)\boldsymbol{\xi}_1\right) \ ,
\end{aligned} \tag{6.35}$$

where $\boldsymbol{\xi}_1\nabla^2\mathbf{F}_0(\boldsymbol{\xi}_0)\boldsymbol{\xi}_1$ is understood as

$$\begin{bmatrix} \sum_{k,l}\xi_k^{(1)}\frac{\partial^2\mathbf{F}_1^{(0)}}{\partial\xi_k^{(1)}\partial\xi_l^{(1)}}\xi_l^{(1)} \\ \sum_{k,l}\xi_k^{(1)}\frac{\partial^2\mathbf{F}_2^{(0)}}{\partial\xi_k^{(1)}\partial\xi_l^{(1)}}\xi_l^{(1)} \\ \sum_{k,l}\xi_k^{(1)}\frac{\partial^2\mathbf{F}_3^{(0)}}{\partial\xi_k^{(1)}\partial\xi_l^{(1)}}\xi_l^{(1)} \end{bmatrix} = \boldsymbol{\xi}^{(1)}\nabla^2\mathbf{F}^{(0)}\boldsymbol{\xi}^{(1)} = \boldsymbol{\xi}_1\nabla^2\mathbf{F}_0(\boldsymbol{\xi}_0)\boldsymbol{\xi}_1 \, .$$

Then, by comparison of the coefficients of identical powers of J_2 on both sides of (6.35), the following sequence of equations is obtained

$$\begin{aligned}
\mathbf{p} &= \mathbf{F}_0(\boldsymbol{\xi}_0) \\
\mathbf{0} &= \nabla\mathbf{F}_0(\boldsymbol{\xi}_0)\boldsymbol{\xi}_1 + \mathbf{F}_1(\boldsymbol{\xi}_0) \\
\mathbf{0} &= \nabla\mathbf{F}_0(\boldsymbol{\xi}_0)\boldsymbol{\xi}_2 + \frac{1}{2}\boldsymbol{\xi}_1\nabla^2\mathbf{F}_0(\boldsymbol{\xi}_0)\boldsymbol{\xi}_1 + \nabla\mathbf{F}_1(\boldsymbol{\xi}_0)\boldsymbol{\xi}_1 \\
\mathbf{0} &= \dots \ ,
\end{aligned} \tag{6.36}$$

which, after some computations, leads to the final result

$$\begin{aligned}
\boldsymbol{\xi}_1 &= \frac{-3R^2}{4\sqrt{GM}\|\mathbf{p}\|^{5/2}}\begin{bmatrix} p_1(p_1^2+p_2^2+2p_3^2) \\ p_2(p_1^2+p_2^2+2p_3^2) \\ -p_3(3p_1^2+3p_2^2+2p_3^2) \end{bmatrix} \\
\boldsymbol{\xi}_2 &= \frac{9R^4}{32GM^{3/2}\|\mathbf{p}\|^{7/2}}\begin{bmatrix} p_1(5p_1^4+5p_2^4-4p_2^2p_3^2-60p_3^4+2p_1^2(5p_2^2-2p_3^2)) \\ p_2(5p_1^4+5p_2^4-4p_2^2p_3^2-60p_3^4+2p_1^2(5p_2^2-2p_3^2)) \\ p_3(-51p_1^4-51p_2^4+20p_2^2p_3^2+20p_3^4+p_1^2(-102p_2^2+20p_3^2)) \end{bmatrix} \ .
\end{aligned} \qquad \diamondsuit$$

The series representation of $\boldsymbol{\xi}(\mathbf{p}, J_2)$, according to Lemma 32, is limited up to and including terms of power J_2^2. The reason is that by definition J_2 is a small number in the order of 10^{-3}. Therefore, tolerating an error smaller than 10^{-6}, when neglecting cubed order terms in J_2, seems reasonable. However, in case the resulting accuracy is insufficient, it is certainly possible to extend the expansion and to solve for the higher order contributions by continuing the solution sequence (6.36).

As usual, besides the considered transformation of the ordinary space position vector $\mathbf{x}$ into the auxiliary position vector $\boldsymbol{\xi}$, cf. Lemma 32, a change in the underlying potential functionals is involved as well. This subject, together with a discussion of the basic features of the proposed transformation, is presented in the next section.

6-1.2 Principal transformation characteristics

Prior to defining the relationship to determine the adjoint potential from the gravitational potential, the idea of an *ellipsoidal regular gravity space* is first formally introduced. Hence, as a result of the last lemma, it is now possible to postulate the following:

Definition 27 *The image of the Earth's exterior, that is* $\mathbf{x} \in ext\ \sigma$*, under the mapping constituted by the series expansion formulae given within the scope of Lemma 32, i.e.*

$$\begin{aligned} \boldsymbol{\xi} = [\xi_i] &= \boldsymbol{\xi}_0(\mathbf{p}) + J_2\boldsymbol{\xi}_1(\mathbf{p}) + J_2^2\boldsymbol{\xi}_2(\mathbf{p}) + \mathrm{O}\big(J_2^3\big) \\ &= \boldsymbol{\xi}_0(\nabla V(\mathbf{x})) + J_2\boldsymbol{\xi}_1(\nabla V(\mathbf{x})) + J_2^2\boldsymbol{\xi}_2(\nabla V(\mathbf{x})) + \mathrm{O}\big(J_2^3\big)\ , \end{aligned} \tag{6.37}$$

is identified as ellipsoidal regular gravity space. In the auxiliary space, the components ξ_i *of the vector* $\boldsymbol{\xi}$ *form the independent Cartesian coordinates.*

And, as seen before, the restriction of the vector $\mathbf{x}$ on the right hand side of (6.37) onto the surface of the Earth, i.e. $\mathbf{x}\big|_\sigma$ leads immediately to a second definition:

Definition 28 *The position vector* $\boldsymbol{\xi}\big|_\Sigma$*, identified as the image of the Earth's surface* σ *under the mapping given by Definition 27, i.e.*

$$\boldsymbol{\xi}\big|_\Sigma = \boldsymbol{\xi}_0(\nabla V(\mathbf{x}\big|_\sigma)) + J_2\boldsymbol{\xi}_1(\nabla V(\mathbf{x}\big|_\sigma)) + J_2^2\boldsymbol{\xi}_2(\nabla V(\mathbf{x}\big|_\sigma)) + \mathrm{O}\big(J_2^3\big)\ , \tag{6.38}$$

constitutes the known surface Σ*, which serves as the boundary surface in ellipsoidal regular gravity space.*

In addition to the above transformation formulae on the level of positions, the gravitational potential also changes, as is known by now, into the associated auxiliary potential, that is the adjoint potential. By definition, the previous expression to obtain the adjoint potential from the gravitational potential, cf. (5.2), remains unaltered, that is

$$\psi(\boldsymbol{\xi}) := \mathbf{x}^\top \nabla V(\mathbf{x}) - V(\mathbf{x}) \quad ; \quad \mathbf{x} = \mathbf{x}(\boldsymbol{\xi})\,. \tag{6.39}$$

In comparison with (5.2) it is worth pointing out that due to the modified transformation (6.37) the resulting position vector $\boldsymbol{\xi}$, which enters into (6.39), is numerically different from the coordinate vector $\boldsymbol{\xi}$, which applies to (5.2). Hence, despite an obvious equivalence of (5.2) and (6.39), the corresponding adjoint potentials differ in terms of actual values from each other.

At last, it should be confirmed that the ellipsoidal gravity space approach indeed possesses the predefined identical mapping property in the case that the gravitational potential V is reduced to the reference potential V_0^{ell} according to (6.8). For that purpose, the situation for the former spherical scenario is briefly recalled in the first instance. As already stated in the context of Lemma 16, the regular gravity space transformation discussed in the last chapter has the following remarkable property. That is, in view of

$$\mathbf{p} = \nabla V(\mathbf{x})$$

and particularly in the case of adopting a spherical approximation for the true potential, i.e.

$$\tilde{\mathbf{p}} = \nabla V_0(\mathbf{x}) = \nabla\left(\frac{GM}{\|\mathbf{x}\|}\right) = -\frac{GM}{\|\mathbf{x}\|^3}\mathbf{x}\,, \tag{6.40}$$

it holds

$$\boldsymbol{\xi}(\tilde{\mathbf{p}}) = \mathbf{x}\,. \tag{6.41}$$

As a matter of fact the new ellipsoidal regular gravity space formulation, Lemma 32, is set up in exactly such a manner that the same relationship is true:

Lemma 33 *Hence, introducing*

$$\tilde{\mathbf{p}} := \nabla V_0^{ell} = \nabla\left(\frac{GM}{\|\mathbf{x}\|}\left(1 - J_2\left(\frac{R}{\|\mathbf{x}\|}\right)^2 P_2(\frac{x_3}{\|\mathbf{x}\|})\right)\right) \tag{6.42}$$

in (6.37) yields the identity mapping scenario

$$\boldsymbol{\xi}(\tilde{\mathbf{p}}, J_2) = \mathbf{x}\,. \tag{6.43}$$

Proof. Let

$$\boldsymbol{\eta} = \left[\begin{array}{cccc} \tilde{p}_1 & \tilde{p}_2 & \tilde{p}_3 & J_2 \end{array}\right]^\top \tag{6.44}$$

be subject to

$$\tilde{p}_i = \frac{\partial}{\partial x_i}\left(\frac{GM}{\|\mathbf{x}\|}\left(1 - J_2\left(\frac{R}{\|\mathbf{x}\|}\right)^2 P_2(\frac{x_3}{\|\mathbf{x}\|})\right)\right) \qquad i = 1, 2, 3\,. \tag{6.45}$$

Now, since the implicit function theorem guarantees the existence of a uniquely defined function

$$\boldsymbol{\xi} = \boldsymbol{\xi}(\boldsymbol{\eta}) \tag{6.46}$$

in case the function $\boldsymbol{\xi}(\boldsymbol{\eta})$ can only be specified in an implicit manner as follows

$$\boldsymbol{\varrho}\left(\boldsymbol{\eta}, \boldsymbol{\xi}(\boldsymbol{\eta})\right) = \mathbf{p} - \nabla_\xi\left(\frac{GM}{\|\boldsymbol{\xi}\|}\left(1 - J_2\left(\frac{R}{\|\boldsymbol{\xi}\|}\right)^2 P_2(\frac{\xi_3}{\|\boldsymbol{\xi}\|})\right)\right) = \mathbf{0}\,, \tag{6.47}$$

then the identical mapping property for the ellipsoidal reference potential is simply obtained by comparison after insertion of (6.44) into (6.47), i.e.

$$\begin{aligned} \mathbf{0} &= \tilde{\mathbf{p}} - \nabla_\xi\left(\frac{GM}{\|\boldsymbol{\xi}\|}\left(1 - J_2\left(\frac{R}{\|\boldsymbol{\xi}\|}\right)^2 P_2(\frac{\xi_3}{\|\boldsymbol{\xi}\|})\right)\right) \\ &= \nabla\left(\frac{GM}{\|\mathbf{x}\|}\left(1 - J_2\left(\frac{R}{\|\mathbf{x}\|}\right)^2 P_2(\frac{x_3}{\|\mathbf{x}\|})\right)\right) - \nabla_\xi\left(\frac{GM}{\|\boldsymbol{\xi}\|}\left(1 - J_2\left(\frac{R}{\|\boldsymbol{\xi}\|}\right)^2 P_2(\frac{\xi_3}{\|\boldsymbol{\xi}\|})\right)\right)\,, \end{aligned} \tag{6.48}$$

which inevitably results in

$$\boldsymbol{\xi} = \mathbf{x}\,, \qquad \diamondsuit \tag{6.49}$$

in order to satisfy the mandatory identity (6.48).

Thus, it can at last be concluded that the corresponding gravity space mapping scenario resembles the situation illustrated in Fig. 5.2. As expected, the presently considered problem remains an exterior BVP. The only difference between the spherical theory presented in the context of the last chapter and the ellipsoidal approach investigated here is that the boundary surface Σ in ellipsoidal regular gravity space is approaching the surface of the Earth σ from a pure geometrical point of view. This aspect will also be approved numerically in the next chapter.
Furthermore, by comparison of (6.37) and (6.39) with (5.1) and (5.2), accounting especially for (6.33), and together with the aforementioned analogy of the resulting mapping scenarios, it can be safely stated that the newly proposed ellipsoidal approach exhibits a similar asymptotic behaviour as discussed previously in Section 5-1.1 in conjunction with the spherical regular gravity space theory. Hence, the ellipsoidal gravity space theory indeed establishes a regular methodology, that is an approach in gravity space free of any singularities.

6-1.3 Contact transformation representation

In order to complete the ellipsoidal regular gravity space approach in view of a contact transformation examination, a third relation, besides the transformation (6.37) of the old momentum vector $\mathbf{p}$ into the new coordinate vector $\boldsymbol{\xi}$ and the transformation (6.39) from the gravitational potential V to the adjoint potential ψ, has to be taken into account. That is, the relationship between the coordinate vector $\mathbf{x}$ and the new momentum vector $\boldsymbol{\pi}$ is of major interest and will be investigated in the present section, thereby referring to the theory already developed in the context of Section 5-1.3. The aspect of determining this relation that still needs to be adressed is treated in the first two consecutive lemmata. Thereafter, based on the completed set of formulae, proof is given that these three relations together constitute a contact transformation, cf. Theorem 9.
The second part of this section is devoted to the problem of finding the inverse relation concerning (6.39), that is the relationship to compute the gravitational potential V from the corresponding adjoint potential ψ. A final lemma addresses this issue.

In order to get started, it is useful to recall the transformation formulae elaborated in the last chapter. The contact transformation representation of regular gravity space approach based on a spherical configuration has been given by the familiar set of equations, see e.g. (5.45)-(5.47),

$$\boldsymbol{\xi} = -\sqrt{GM}\frac{\mathbf{p}}{\|\mathbf{p}\|^{3/2}} \tag{6.50}$$

$$\psi = \mathbf{p}^\top\mathbf{x} - V(\mathbf{x}) \tag{6.51}$$

$$\boldsymbol{\pi} = \boldsymbol{\alpha}\mathbf{x}\,, \tag{6.52}$$

subject to the old momentum vector

$$\mathbf{p} = \nabla V(\mathbf{x})\,. \tag{6.53}$$

Right now the status quo is as follows. The coordinate mapping defined by (6.50) is substituted by the revised ellipsoidal mapping formulae (6.37). On the other hand, since the potential relation (6.51) is left unaltered, see (6.39), the third relation (6.52) has to be modified as well in order to satisfy the necessary identity (2.83) to preserve the contact transformation property. The new transformation corresponding to (6.52) can be found by the reasoning given in the lemma following next, which is in fact closely related to Lemma 20 established previously within the framework of deriving (6.52):

Lemma 34 *The relation to determine the momentum vector* $\boldsymbol{\pi} = \nabla_\xi\psi(\boldsymbol{\xi})$ *from the position vector* $\mathbf{x}$ *is given by*

$$\boldsymbol{\pi} = \bar{\boldsymbol{\alpha}}\mathbf{x}\,, \tag{6.54}$$

with the matrix $\bar{\boldsymbol{\alpha}}$ *specified according to*

$$\bar{\boldsymbol{\alpha}} = \left[\frac{\partial\mathbf{p}}{\partial\boldsymbol{\xi}}\right]. \tag{6.55}$$

Proof. Starting from the definition for the new momentum vector $\boldsymbol{\pi} = \nabla_\xi\psi(\boldsymbol{\xi})$ given in index notation

$$\pi_i = \frac{\partial\psi}{\partial\xi_i}\,, \tag{6.56}$$

then insertion of the expression (6.39) given for the adjoint potential and application of the chain rule of differential calculus results in, see also (5.35),

$$\begin{aligned}
\pi_i &= \frac{\partial}{\partial\xi_i}(p_k x_k - V) \\
&= \frac{\partial p_k}{\partial\xi_i}x_k + p_k\frac{\partial x_k}{\partial\xi_i} - \frac{\partial V}{\partial\xi_i} \\
&= \frac{\partial p_k}{\partial\xi_i}x_k + p_k\frac{\partial x_k}{\partial\xi_i} - \frac{\partial V}{\partial x_k}\frac{\partial x_k}{\partial\xi_i} \\
&= \frac{\partial p_k}{\partial\xi_i}x_k + p_k\frac{\partial x_k}{\partial\xi_i} - p_k\frac{\partial x_k}{\partial\xi_i} \\
&= \frac{\partial p_k}{\partial\xi_i}x_k\,.
\end{aligned} \tag{6.57}$$

Now, identifying

$$\left[\frac{\partial\mathbf{p}}{\partial\boldsymbol{\xi}}\right] = \bar{\boldsymbol{\alpha}}$$

in (6.57), yields

$$\boldsymbol{\pi} = \bar{\boldsymbol{\alpha}}\mathbf{x}\,. \qquad \diamondsuit$$

In contrast to Lemma 20, where the matrix $\boldsymbol{\alpha}$ has been explicitly introduced by means of formally differentiating the function $\mathbf{p}(\boldsymbol{\xi})$, i.e. (5.33), with respect to $\boldsymbol{\xi}$, the matrix $\bar{\boldsymbol{\alpha}}$ is for the sake of conciseness not worked out in detail here. It should be pointed out however, that from a theoretical point of view it is not a problem to evaluate (6.55) by differentiating the corresponding expression $\mathbf{p}(\boldsymbol{\xi})$, that is (6.26), with respect to $\boldsymbol{\xi}$. Hence, similarly to (5.30), it can be recorded that the matrix $\bar{\boldsymbol{\alpha}}$ results as a function of $\boldsymbol{\xi}$

$$\bar{\boldsymbol{\alpha}} = \bar{\boldsymbol{\alpha}}(\boldsymbol{\xi})\,. \tag{6.58}$$

On the other hand, when the coordinate vector $\boldsymbol{\xi}$ in view of (6.53) is considered as a function of the coordinate vector $\mathbf{x}$, that is $\boldsymbol{\xi} = \boldsymbol{\xi}(\mathbf{x})$, the question about the inverse mapping $\mathbf{x} = \mathbf{x}(\boldsymbol{\xi})$ arises. The next lemma is concerned with finding the answer:

Lemma 35 *Inversion of (6.54) yields*

$$\mathbf{x} = \bar{\boldsymbol{\gamma}}\boldsymbol{\pi}\,, \tag{6.59}$$

subject to the matrix

$$\bar{\boldsymbol{\gamma}} = \left[\frac{\partial \boldsymbol{\xi}}{\partial \mathbf{p}}\right]. \tag{6.60}$$

Proof. Inversion of (6.54), thereby taking (6.55) into account, leads directly to

$$\begin{aligned} \mathbf{x} &= \bar{\boldsymbol{\alpha}}^{-1}\boldsymbol{\pi} \\ &= \left[\frac{\partial \mathbf{p}}{\partial \boldsymbol{\xi}}\right]^{-1}\boldsymbol{\pi}\,. \end{aligned} \tag{6.61}$$

Analogously to Lemma 21 and in conformity with Lemma 1, the following matrix identity holds between the corresponding Jacobian matrices of the transformation $\mathbf{p} = \mathbf{p}(\boldsymbol{\xi})$ and its associated inverse transformation $\boldsymbol{\xi} = \boldsymbol{\xi}(\mathbf{p})$

$$\left[\frac{\partial \mathbf{p}}{\partial \boldsymbol{\xi}}\right]^{-1} = \left[\frac{\partial \boldsymbol{\xi}}{\partial \mathbf{p}}\right]. \tag{6.62}$$

Consequently, using (6.62), (6.61) becomes

$$\mathbf{x} = \left[\frac{\partial \boldsymbol{\xi}}{\partial \mathbf{p}}\right]\boldsymbol{\pi}$$

or by introducing the matrix $\bar{\boldsymbol{\gamma}}$ as follows

$$\bar{\boldsymbol{\gamma}} = \left[\frac{\partial \boldsymbol{\xi}}{\partial \mathbf{p}}\right],$$

it applies equivalently

$$\mathbf{x} = \bar{\boldsymbol{\gamma}}\boldsymbol{\pi}\,. \qquad \diamondsuit$$

As desired, by means of (6.59) an elegant way is found to express the inverse transformation of $\boldsymbol{\xi} = \boldsymbol{\xi}(\mathbf{x})$. The fact that (6.59) indeed represents the sought-after function $\mathbf{x} = \mathbf{x}(\boldsymbol{\xi})$, can be understood against the background of (6.56) as well as (6.58) and (6.61). More precisely, in view of (6.56), the momentum vector $\boldsymbol{\pi}$ can be considered as a function of $\boldsymbol{\xi}$, that is $\boldsymbol{\pi} = \nabla_{\xi}\psi(\boldsymbol{\xi}) = \boldsymbol{\pi}(\boldsymbol{\xi})$. In addition, from $\bar{\boldsymbol{\alpha}} = \bar{\boldsymbol{\alpha}}(\boldsymbol{\xi})$, cf. (6.58), results according to (6.61) that the matrix $\bar{\boldsymbol{\gamma}}$, determined as the inverse of $\bar{\boldsymbol{\alpha}}$, can be considered as a function of $\boldsymbol{\xi}$ as well, i.e.

$$\bar{\boldsymbol{\gamma}} = \bar{\boldsymbol{\gamma}}(\boldsymbol{\xi})\,. \tag{6.63}$$

Consequently, (6.59) can also be looked at in the following way

$$\mathbf{x} = \bar{\boldsymbol{\gamma}}(\boldsymbol{\xi})\boldsymbol{\pi}(\boldsymbol{\xi}) = \mathbf{x}(\boldsymbol{\xi})\,. \tag{6.64}$$

In a similar manner as has been done in Lemma 20 and 21, $\bar{\boldsymbol{\alpha}} = \bar{\boldsymbol{\alpha}}(\mathbf{p})$ as well as $\bar{\boldsymbol{\gamma}} = \bar{\boldsymbol{\gamma}}(\mathbf{p})$ can be deduced by the application of (6.37) to (6.58) and (6.63) respectively.

This sets the stage for the following theorem identifying the formulae for the ellipsoidal regular gravity space approach, i.e.

$$\boldsymbol{\xi} = \boldsymbol{\xi}_0(\mathbf{p}) + J_2\boldsymbol{\xi}_1(\mathbf{p}) + J_2^2\boldsymbol{\xi}_2(\mathbf{p}) + \mathrm{O}\big(J_2^3\big) \tag{6.65}$$

$$\psi = \mathbf{p}^{\top}\mathbf{x} - V(\mathbf{x}) \tag{6.66}$$

$$\boldsymbol{\pi} = \bar{\boldsymbol{\alpha}}\mathbf{x}\,, \tag{6.67}$$

as a contact transformation, compare also Theorem 6, and completes this first part:

Theorem 9 *The transformations (6.65)-(6.67) are in agreement with Definition 2 and consequently constitute a contact transformation.*

Proof. Simple calculations based on the accomplishments achieved in this section verify that (6.65)-(6.67) indeed satisfy the required condition (2.83) in order to establish a contact transformation

$$\begin{aligned} d\psi - d\xi_k\pi_k &= d(p_i x_i - V) - d\xi_k\pi_k \\ &= x_i dp_i + p_i dx_i - dV - \frac{\partial \xi_k}{\partial p_n}dp_n\bar{\alpha}_{ki}x_i \\ &= x_i dp_i + p_i dx_i - dV - \bar{\gamma}_{kn}dp_n\bar{\alpha}_{ki}x_i \\ &= -\left(dV - dx_i p_i\right)\,. \qquad \diamondsuit \end{aligned}$$

As indicated before, after having provided the forward and backward coordinate transformations, that is $\boldsymbol{\xi} = \boldsymbol{\xi}(\mathbf{x})$ and $\mathbf{x} = \mathbf{x}(\boldsymbol{\xi})$, the only open assignment remaining is to find the inverse transformation of (6.66). For that purpose, a second theorem looks into that subject:

Theorem 10 *The inverse transformation of (6.39) reads as*

$$V(\mathbf{x}) = \mathbf{q}(\boldsymbol{\xi})^\top \nabla_\xi \psi(\boldsymbol{\xi}) - \psi(\boldsymbol{\xi}) \quad ; \quad \boldsymbol{\xi} = \boldsymbol{\xi}(\mathbf{x}) \,, \tag{6.68}$$

subject to the vector

$$\mathbf{q}(\boldsymbol{\xi})^\top = \mathbf{p}(\boldsymbol{\xi})^\top \bar{\boldsymbol{\alpha}}(\boldsymbol{\xi})^{-1} \,. \tag{6.69}$$

Proof. To begin with, the relation to determine the adjoint potential ψ from quantities given solely in geometry space, i.e. (6.66),

$$\psi(\boldsymbol{\xi}) = \mathbf{p}(\mathbf{x})^\top \mathbf{x} - V(\mathbf{x}) \tag{6.70}$$

is rearranged as follows

$$V(\mathbf{x}) = \mathbf{p}(\boldsymbol{\xi})^\top \mathbf{x}(\boldsymbol{\xi}) - \psi(\boldsymbol{\xi}) \,. \tag{6.71}$$

With respect to (6.71) the following fundamental remarks are of importance. At first, comparable to (6.70), equation (6.71) is meant for calculating the gravitational potential V from quantities given exclusively in auxiliary space. Hence, in (6.70), $\mathbf{p}(\boldsymbol{\xi})$ applies according to (6.26). Moreover, instead of $\mathbf{x}(\boldsymbol{\xi})$, the corresponding expression given in (6.64) enters into (6.71), which allows (6.71) to be rewritten in the following way

$$V(\mathbf{x}) = \mathbf{p}(\boldsymbol{\xi})^\top \bar{\boldsymbol{\gamma}}(\boldsymbol{\xi}) \boldsymbol{\pi}(\boldsymbol{\xi}) - \psi(\boldsymbol{\xi}) \,. \tag{6.72}$$

Now, further modification of (6.72) is accomplished by means of defining

$$\mathbf{q}(\boldsymbol{\xi}) := \mathbf{p}(\boldsymbol{\xi})^\top \bar{\boldsymbol{\gamma}}(\boldsymbol{\xi}) = \mathbf{p}(\boldsymbol{\xi})^\top \bar{\boldsymbol{\alpha}}(\boldsymbol{\xi})^{-1} \tag{6.73}$$

and taking

$$\boldsymbol{\pi}(\boldsymbol{\xi}) = \nabla_\xi \psi(\boldsymbol{\xi}) \tag{6.74}$$

into consideration, which leads at length to

$$V(\mathbf{x}) = \mathbf{q}(\boldsymbol{\xi})^\top \nabla_\xi \psi(\boldsymbol{\xi}) - \psi(\boldsymbol{\xi}) \quad ; \quad \boldsymbol{\xi} = \boldsymbol{\xi}(\mathbf{x}) \,. \qquad \diamondsuit$$

Application of $\bar{\boldsymbol{\alpha}}^{-1}$ in (6.69) instead of $\bar{\boldsymbol{\gamma}}$, cf. (6.73), accounts for the fact that in order to derive $\bar{\boldsymbol{\gamma}} = \left[\frac{\partial \boldsymbol{\xi}}{\partial \mathbf{p}}\right]$, it is necessary for computational reasons to exercise the option (6.62) and derive $\bar{\boldsymbol{\alpha}} = \left[\frac{\partial \mathbf{p}}{\partial \boldsymbol{\xi}}\right]$ in the first instance and then perform a matrix inversion thereafter. The reason is that it was not possible to find a closed solution for $\boldsymbol{\xi}(\mathbf{p})$.

Remark 27 It is worthwhile to draw a comparison between (5.18) and (6.68). It is quite obvious that in the case of the regular gravity space approach presented in the last chapter, the vector $\mathbf{q}(\boldsymbol{\xi})$ reduces to $-\frac{1}{2}\boldsymbol{\xi}$, which can be easily verified by inspecting (5.67). Of course this special case simplifies matters considerably as becomes clear in the further course of this chapter, when the more complex form $\mathbf{q}(\boldsymbol{\xi})$ has to be accounted for. Concerning the ellipsoidal regular gravity space approach, the vector $\mathbf{q}(\boldsymbol{\xi})$ has been explicitly derived using a computer algebra system, such as e.g. *Mathematica*, and is stated in Appendix B-1.

As performed in Section 4-3 and Section 5-1 in a likewise manner, and after having introduced all transformation formulae and their associated characteristics, the ellipsoidal regular gravity space counterpart of the GBVP, given according to Definition 13, will be establish in the next section.

6-2 The nonlinear GBVP in ellipsoidal regular gravity space

In order to derive the nonlinear GBVP in ellipsoidal regular gravity space, this section aims to embark on the same strategy as previously employed in Section 5-2. See Lemma 24 to Lemma 27 and (5.66) in particular. Hence, based on the spherical scenario, the nonlinear BVP in regular gravity space, Definition 22, is first recalled for the purpose of comparison. That is, the adjoint potential ψ was found to solve the following field equation

$$(\operatorname{tr} \boldsymbol{\Phi})^2 - \operatorname{tr} \boldsymbol{\Phi}^2 = 0, \quad \boldsymbol{\xi} \in ext\, \Sigma \tag{6.75}$$

under the boundary condition

$$-\left(\frac{1}{2}\boldsymbol{\xi}^\top \nabla\psi + \psi\right)\Big|_\Sigma = v\,, \tag{6.76}$$

which is subject to the gravimetric telluroid Σ and the functional matrix

$$\boldsymbol{\Phi} = [\Phi_{ij}] = \left[\gamma_{in}\gamma_{jm}\frac{\partial^2\psi}{\partial\xi_n\partial\xi_m} - \beta_{imj}\frac{\partial\psi}{\partial\xi_m}\right]\,, \tag{6.77}$$

with the known functions γ_{kl}, β_{pqr} given in Lemma 21 and Lemma 25.

Now, following the line of argument described earlier in Section 5-2 to derive the above relations, a first lemma, cf. also Lemma 24, introduces the following new quantity:

Lemma 36 *Define the coefficients* $\bar\beta_{imj}$ *by*

$$\bar\beta_{imj} := \frac{\partial\bar\gamma_{im}}{\partial p_j} \tag{6.78}$$

then

$$\left[\frac{\partial^2 V}{\partial x_i\partial x_j}\right]^{-1} = \left[\bar\gamma_{il}\bar\gamma_{jm}\frac{\partial^2\psi}{\partial\xi_l\partial\xi_m} + \bar\beta_{imj}\frac{\partial\psi}{\partial\xi_m}\right] \tag{6.79}$$

holds.

Proof. Again, simple calculations based on the chain rule of differential calculus lead to

$$\left[\frac{\partial\xi_i}{\partial x_k}\right] = \left[\frac{\partial\xi_i}{\partial p_l}\frac{\partial p_l}{\partial x_k}\right] = \left[\bar\gamma_{il}\frac{\partial^2 V}{\partial x_l\partial x_k}\right]$$

and when the inverse relation is considered,

$$\begin{aligned}\left[\frac{\partial x_k}{\partial\xi_j}\right] = \left[\frac{\partial(\bar\gamma_{km}\pi_m)}{\partial\xi_j}\right] &= \left[\bar\gamma_{km}\frac{\partial^2\psi}{\partial\xi_m\partial\xi_j} + \frac{\partial\bar\gamma_{km}}{\partial p_n}\frac{\partial p_n}{\partial\xi_j}\frac{\partial\psi}{\partial\xi_m}\right]\\ &= \left[\bar\gamma_{km}\frac{\partial^2\psi}{\partial\xi_m\partial\xi_j} + \frac{\partial\bar\gamma_{km}}{\partial p_n}\bar\alpha_{nj}\frac{\partial\psi}{\partial\xi_m}\right]\end{aligned}$$

applies. Hence, the associated matrix product yields the unit matrix

$$\begin{aligned}[\delta_{ij}] = \left[\frac{\partial x_i}{\partial\xi_k}\frac{\partial\xi_k}{\partial x_j}\right] &= \left[\bar\gamma_{im}\frac{\partial^2\psi}{\partial\xi_m\partial\xi_k} + \frac{\partial\bar\gamma_{im}}{\partial p_n}\bar\alpha_{nk}\frac{\partial\psi}{\partial\xi_m}\right]\left[\bar\gamma_{kl}\frac{\partial^2 V}{\partial x_l\partial x_j}\right]\\ &= \left[\bar\gamma_{im}\bar\gamma_{lk}\frac{\partial^2\psi}{\partial\xi_m\partial\xi_k} + \frac{\partial\bar\gamma_{im}}{\partial p_l}\frac{\partial\psi}{\partial\xi_m}\right]\left[\frac{\partial^2 V}{\partial x_l\partial x_j}\right].\end{aligned}$$

Finally, rearranging the last relationship results in

$$\left[\frac{\partial^2 V}{\partial x_i\partial x_l}\right]^{-1} = \left[\bar\gamma_{im}\bar\gamma_{lk}\frac{\partial^2\psi}{\partial\xi_m\partial\xi_k} + \bar\beta_{iml}\frac{\partial\psi}{\partial\xi_m}\right]. \qquad \diamondsuit$$

In contrast to Section 5-2, evaluation of $\bar\beta_{imj}$ is set aside for the time being since it will turn out within the ongoing linearization and approximation procedure that an explicit knowledge of $\bar\beta_{imj}$ is not essential. Thus, the next step according to the modus operandi familiar from the previous chapter is to introduce a new matrix $\bar{\boldsymbol{\Phi}}$ in order to find the pendant of Laplace's equation in ellipsoidal regular gravity space. The next lemma concentrates on this task:

Lemma 37 *Let the potential* V *be harmonic and the functional matrix* $\bar{\boldsymbol{\Phi}}$ *be given by*

$$\bar{\boldsymbol{\Phi}} = [\bar\Phi_{ij}] = \left[\bar\gamma_{im}\bar\gamma_{jl}\frac{\partial^2\psi}{\partial\xi_m\partial\xi_l} + \bar\beta_{imj}\frac{\partial\psi}{\partial\xi_m}\right]\,, \tag{6.80}$$

then the adjoint potential ψ *satisfies the following nonlinear field equation in ellipsoidal regular gravity space*

$$\operatorname{tr}\bar{\boldsymbol{\Phi}}^{-1} = 0\,. \tag{6.81}$$

Proof. The matrix identity

$$\mathbf{V} = \bar{\mathbf{\Phi}}^{-1}$$

results as a consequence of Lemma 36 and (6.80). As before, starting from Laplace's equation and by taking (2.31) into account, the following conclusion holds for the corresponding field equation in ellipsoidal regular gravity space

$$\Delta V = \mathrm{tr}\,\mathbf{V} = \mathrm{tr}\,\bar{\mathbf{\Phi}}^{-1} = 0\,. \qquad \diamondsuit$$

Once more, a last lemma to avoid the matrix inverse $\bar{\mathbf{\Phi}}^{-1}$, which occurs in the present form for the field equation, i.e. (6.81), is conducted:

Lemma 38 *The partial differential equation for ψ given in Lemma 37 can be reformulated as follows*

$$\left(\mathrm{tr}\,\bar{\mathbf{\Phi}}\right)^2 - \mathrm{tr}\,\bar{\mathbf{\Phi}}^2 = 0\,. \tag{6.82}$$

Proof. In analogy to Lemma 11 and Lemma 27, the proof by application of Cramer's rule is skipped again.

Once more, in addition to the above considerations establishing the field equation applicable in ellipsoidal regular gravity space, determination of the corresponding boundary condition is of equal interest and will be accounted for in the following. To this end, (6.68) is restricted onto the relevant boundary surfaces, cf. (5.66), which leads to

$$V\big|_\sigma = \left(\mathbf{q}^\top \nabla_\xi \psi - \psi\right)\big|_\Sigma\,. \tag{6.83}$$

Thus, on the basis of (6.82) and (6.83) the designated BVP for the adjoint potential in the context of the new ellipsoidal approach can be specified as follows:

Definition 29 *The geodetic boundary value problem in ellipsoidal regular gravity space concerns the following problem: the data, i.e. gravitational potential values, are now given at the known surface Σ*

$$v : \Sigma \rightarrow \mathbb{R}$$

and required to be found is a real function $\psi(\boldsymbol{\xi}) : ext\ \Sigma \rightarrow \mathbb{R}$

$$\begin{aligned} (\mathrm{tr}\,\bar{\mathbf{\Phi}})^2 - \mathrm{tr}\,\bar{\mathbf{\Phi}}^2 &= 0\,, \quad \boldsymbol{\xi} \in ext\ \Sigma && (6.84)\\ \left(\mathbf{q}^\top \nabla_\xi \psi - \psi\right)\big|_\Sigma &= v\,, && (6.85)\end{aligned}$$

which is the solution of the second-order partial differential equation (6.84) under the boundary condition (6.85), with $\bar{\mathbf{\Phi}}$ given according to (6.80) and $\mathbf{q}$ defined by (6.73).

Similarly to Definition 22, (6.84) and (6.85) describe a nonlinear oblique exterior BVP, thereby establishing the GBVP in ellipsoidal regular gravity space. The field equation (6.84), apart from the coefficients $\bar{\gamma}_{ij}$ and $\bar{\beta}_{imj}$ associated with $\bar{\mathbf{\Phi}}$, is resembling a Monge-Ampère type of differential equation. And as far as the boundary condition is concerned, it can be stated that in contrast to the GBVP in geometry space, the GBVP according to Definition 29 represents a BVP with a fixed boundary, since Σ in (6.85) is a known surface by means of (6.38). Moreover, the boundary condition (6.85) remains linear despite its modified appearance pointed out in Remark 27.

Again, comparable to (5.70), the Earth's surface σ is obtained from the solution for $\psi(\boldsymbol{\xi})$ according to Lemma 35

$$\begin{aligned} \mathbf{x}\big|_\sigma &= \bar{\gamma}(\boldsymbol{\xi}\big|_\Sigma)\boldsymbol{\pi}(\boldsymbol{\xi}\big|_\Sigma)\\ &= \bar{\gamma}(\boldsymbol{\xi}\big|_\Sigma)\nabla_\xi \psi(\boldsymbol{\xi}\big|_\Sigma)\,. \end{aligned} \tag{6.86}$$

Remark 28 At last, a comparison of the BVP given in the context of Definition 22 and the BVP defined above reveals a high degree of conformance in spite of the varying boundary conditions. In fact, substitution of the spherical reference potential by an ellipsoidal reference potential is predominantly reflected in an altered transformation behaviour between ordinary and auxiliary space, cf. (5.1) and (6.37). This means that in lieu of former quantities such as $\boldsymbol{\alpha}$, $\boldsymbol{\gamma}$, β_{imj} and $\mathbf{\Phi}$, occurring in the previous version of the GBVP given by (6.75)-(6.77), the same quantities apply again in the present context, though characterized with an overline to account for the revised ellipsoidal transformation formulae. However, in terms of the resulting BVP, it can be concluded that the overall mathematical structure remains the same. Consequently, the further proceeding is performed as normal and the usual linearization step is considered next.

6-3 Linearization of the GBVP in ellipsoidal regular gravity space

In Section 3-4, Section 4-5 and Section 5-3 the necessity and the modus operandi for linearizing the respective nonlinear problem as supplied before for each case has been extensively examined. Consequently, it is the aim to treat the nonlinear problem associated with the new ellipsoidal configuration as introduced according to Definition 29 in the approved manner. Only this time an *ellipsoidal linearization point* is chosen. That is, in the linearization step the true adjoint potential is approximated by means of an *adjoint ellipsoidal normal potential*, which results as the auxiliary space counterpart of the gravitational normal potential V_0^{ell} given in (6.7). Thus, by separating the true adjoint potential ψ into a *normal* part and a *disturbing* part and by inserting the resulting sum of *adjoint normal potential* $\bar{\psi}_0$ and *adjoint disturbing potential* $\delta\bar{\psi}$ into the nonlinear field equation (6.84), thereby neglecting all terms of order two and higher, the corresponding partial differential equation in linear approximation for the adjoint disturbing potential $\delta\bar{\psi}$ is found. Again, the associated boundary condition is elaborated thereafter.

To begin with, the first theorem establishes the adjoint normal potential $\bar{\psi}_0$, which corresponds to the normal gravitational potential V_0^{ell} specified in (6.7). Note that, whereas the adjoint spherical normal potential $\bar{\psi}_0$, (5.72), was identified as a first order approximation of the true adjoint potential ψ, the adjoint ellipsoidal normal potential $\bar{\psi}_0$ represents a second and consequently higher order approximation of the true adjoint potential ψ. Next, in contrast to Section 5-3, the desired linearization of the nonlinear partial differential equation for ψ is directly accomplished in the following lemma by splitting up the adjoint potential in (6.82) once more into its reference and its disturbing component, and by restricting the resulting equation to the linear terms. The reasons for skipping considerations in the present context, as expected in the style of Lemma 28 and Lemma 29, cf. Section 5-3, will become clear in the remark after determination of the field equation for the adjoint disturbing potential $\delta\bar{\psi}$. In addition, after identification of the related field equation, investigations on the corresponding boundary condition are conducted. Finally, the linearized version of the BVP (6.84)-(6.85), given at the end of Section 6-2, concludes this section.

Now, in view of the reasoning given already in the framework of Theorem 7, the adjoint ellipsoidal normal potential, which emanates by definition from the gravitational reference potential V_0^{ell} under the mapping (6.39), is determined as follows:

Theorem 11 *Assume that the gravitational normal potential* V_0^{ell} *is given by*

$$V_0^{ell}(\mathbf{x}) = \frac{GM}{\|\mathbf{x}\|}\left(1 - J_2\left(\frac{R}{\|\mathbf{x}\|}\right)^2 P_2(\frac{x_3}{\|\mathbf{x}\|})\right) , \tag{6.87}$$

see also (6.7), then the corresponding adjoint potential $\bar{\psi}_0$ *is given by*

$$\bar{\psi}_0 = -\frac{2GM}{\|\boldsymbol{\xi}\|}\left(1 - 2J_2\left(\frac{R}{\|\boldsymbol{\xi}\|}\right)^2 P_2(\frac{\xi_3}{\|\boldsymbol{\xi}\|})\right) . \tag{6.88}$$

Proof. Starting from (6.66), i.e.

$$\psi = \mathbf{p}^\top\mathbf{x} - V ,$$

the corresponding adjoint ellipsoidal normal potential $\bar{\psi}_0$ is given by

$$\bar{\psi}_0 = \tilde{\mathbf{p}}^\top\mathbf{x} - V_0^{ell} . \tag{6.89}$$

In addition, because of $\tilde{\mathbf{p}} = \nabla V_0^{ell}$ in (6.89), the relationship $\mathbf{x} = \boldsymbol{\xi}$ holds true, cf. Lemma 33. Hence, $\bar{\psi}_0$ can be obtained by straightforward computations, see also `OUT[5]` in Appendix B-2,

$$\begin{aligned}
\bar{\psi}_0 &= \tilde{\mathbf{p}}^\top\boldsymbol{\xi} - V_0^{ell} \\
&= \nabla_\xi\left(\frac{GM}{\|\boldsymbol{\xi}\|}\left(1 - J_2\left(\frac{R}{\|\boldsymbol{\xi}\|}\right)^2 P_2(\frac{\xi_3}{\|\boldsymbol{\xi}\|})\right)\right)^\top\boldsymbol{\xi} - \frac{GM}{\|\boldsymbol{\xi}\|}\left(1 - J_2\left(\frac{R}{\|\boldsymbol{\xi}\|}\right)^2 P_2(\frac{\xi_3}{\|\boldsymbol{\xi}\|})\right) \\
&= -\frac{2GM}{\|\boldsymbol{\xi}\|^5}\left(\|\boldsymbol{\xi}\|^4 + R^2(\xi_1^2 + \xi_2^2 - 2\xi_3^2)J_2\right) \\
&= -\frac{2GM}{\|\boldsymbol{\xi}\|}\left(1 - 2J_2\left(\frac{R}{\|\boldsymbol{\xi}\|}\right)^2 P_2(\frac{\xi_3}{\|\boldsymbol{\xi}\|})\right) . \qquad \diamondsuit
\end{aligned}$$

As argued above, the next step is to split the unknown adjoint potential ψ into its known normal part $\bar{\psi}_0$ and its disturbing part $\delta\bar{\psi}$

$$\psi = \bar{\psi}_0 + \delta\bar{\psi}\,. \tag{6.90}$$

Accordingly, besides $\bar{\mathbf{\Phi}}$, see also Lemma 37, the corresponding functional matrices will be denoted by $\bar{\mathbf{\Phi}}_0$ and $\delta\bar{\mathbf{\Phi}}$

$$\bar{\mathbf{\Phi}}_0 = \left(\bar{\gamma}_{mi}\bar{\gamma}_{kl}\frac{\partial^2\bar{\psi}_0}{\partial\xi_m\partial\xi_k} + \bar{\beta}_{iml}\frac{\partial\bar{\psi}_0}{\partial\xi_m}\right) \tag{6.91}$$

$$\delta\bar{\mathbf{\Phi}} = \left(\bar{\gamma}_{mi}\bar{\gamma}_{kl}\frac{\partial^2\delta\bar{\psi}}{\partial\xi_m\partial\xi_k} + \bar{\beta}_{iml}\frac{\partial\delta\bar{\psi}}{\partial\xi_m}\right)\,, \tag{6.92}$$

which leads to the following expression

$$\bar{\mathbf{\Phi}} = \bar{\mathbf{\Phi}}_0 + \delta\bar{\mathbf{\Phi}}\,. \tag{6.93}$$

Hence, in order to obtain the desired linear version for the nonlinear GBVP in ellipsoidal regular gravity space, established at the end of Section 6-2, the following theorem is considered:

Theorem 12 *Taking (6.93) into consideration, the nonlinear field equation established by Lemma 38 becomes*

$$\mathrm{tr}\,\bar{\mathbf{\Phi}}_0\mathrm{tr}\,\delta\bar{\mathbf{\Phi}} - \mathrm{tr}\,[\bar{\mathbf{\Phi}}_0\delta\bar{\mathbf{\Phi}}] = \frac{1}{2}\left(\mathrm{tr}\,\bar{\mathbf{\Phi}}_0^2 - \left(\mathrm{tr}\,\bar{\mathbf{\Phi}}_0\right)^2\right)\,, \tag{6.94}$$

when neglecting terms of order $\mathrm{O}\big(\delta\bar{\mathbf{\Phi}}^2\big)$.

Proof. Following the example given in the context of Lemma 30, applying (6.93) to (6.82) leads to

$$\begin{aligned}
0 = \left(\mathrm{tr}\,\bar{\mathbf{\Phi}}\right)^2 - \mathrm{tr}\,\bar{\mathbf{\Phi}}^2 &= \left(\mathrm{tr}\,\bar{\mathbf{\Phi}}_0 + \mathrm{tr}\,\delta\bar{\mathbf{\Phi}}\right)^2 - \mathrm{tr}\,\left[\bar{\mathbf{\Phi}}_0 + \delta\bar{\mathbf{\Phi}}\right]^2 \\
&= \left(\mathrm{tr}\,\bar{\mathbf{\Phi}}_0\right)^2 + 2\mathrm{tr}\,\bar{\mathbf{\Phi}}_0\mathrm{tr}\,\delta\bar{\mathbf{\Phi}} + \left(\mathrm{tr}\,\delta\bar{\mathbf{\Phi}}\right)^2 - \mathrm{tr}\,\bar{\mathbf{\Phi}}_0^2 - 2\mathrm{tr}\,\left[\bar{\mathbf{\Phi}}_0\delta\bar{\mathbf{\Phi}}\right] - \mathrm{tr}\,\delta\bar{\mathbf{\Phi}}^2 \\
&= 2\bigg(\mathrm{tr}\,\bar{\mathbf{\Phi}}_0\mathrm{tr}\,\delta\bar{\mathbf{\Phi}} - \mathrm{tr}\,\left[\bar{\mathbf{\Phi}}_0\delta\bar{\mathbf{\Phi}}\right]\bigg) + \left(\mathrm{tr}\,\bar{\mathbf{\Phi}}_0\right)^2 - \mathrm{tr}\,\bar{\mathbf{\Phi}}_0^2 + \mathrm{O}\big(\delta\bar{\mathbf{\Phi}}^2\big)\,.
\end{aligned} \tag{6.95}$$

Thus, rearranging (6.95) yields

$$\mathrm{tr}\,\bar{\mathbf{\Phi}}_0\mathrm{tr}\,\delta\bar{\mathbf{\Phi}} - \mathrm{tr}\,[\bar{\mathbf{\Phi}}_0\delta\bar{\mathbf{\Phi}}] = \frac{1}{2}\left(\mathrm{tr}\,\bar{\mathbf{\Phi}}_0^2 - \left(\mathrm{tr}\,\bar{\mathbf{\Phi}}_0\right)^2\right)\,. \qquad \diamondsuit$$

The above theorem indicates that for the ellipsoidal gravity space transformation the linearized problem is inhomogeneous, which is in contrast to the spherical gravity space approach, where according to Lemma 30, the linear problem has been found to be homogeneous. The reason is that within the scope of the spherical regular gravity space approach in linear approximation, see also Section 5-3,

$$\left(\mathrm{tr}\,\mathbf{\Phi}_0\right)^2 - \mathrm{tr}\,\mathbf{\Phi}_0^2 = 0 \tag{6.96}$$

holds, whereas obviously

$$\left(\mathrm{tr}\,\bar{\mathbf{\Phi}}_0\right)^2 - \mathrm{tr}\,\bar{\mathbf{\Phi}}_0^2 \neq 0 \tag{6.97}$$

is true in the course of the linearization step for the ellipsoidal approach. Thus, the next questions to be raised are whether the expression $\left(\mathrm{tr}\,\bar{\mathbf{\Phi}}_0\right)^2 - \mathrm{tr}\,\bar{\mathbf{\Phi}}_0^2$ is at the least small of order $\mathrm{O}(J_2)$ and whether the partial differential equation (6.94) can therefore be modified further. Similarly to Theorem 8 and under certain assumptions discussed later on, the adjoint disturbing potential $\delta\bar{\psi}$ is preferably expected to satisfy Laplace's equation. In fact, these questions will be answered affirmatively in the following section.

Remark 29 With regard to (6.96) and (6.97) the following additional considerations are important. First of all, both relations exhibit the same mathematical structure as the corresponding nonlinear field equations (6.75) and (6.82). Furthermore, within the linearization procedure involved in the context of the regular gravity space approach outlined in the last chapter, it has been possible to derive the matrix $\mathbf{\Phi}_0$ explicitly, cf. Lemma 28. As a result, the respective adjoint spherical normal potential ψ_0, see Theorem 7, has been shown in Lemma 29 to satisfy the partial differential equation (5.65), which applies for the nonlinear BVP in regular gravity space. That is, proof has been given for (6.96). Admittedly, proceeding in such a manner in the case under consideration, i.e. evaluation of (6.91) in extenso, would prove very cumbersome and hardly promising and was not considered on this account so far. Hence, in the further course of the investigations it is necessary for the time being to act on the assumption (6.97). As a consequence of this inhomogeneity it is not possible to find a Laplace type of differential equation for the adjoint disturbing potential, at least not within the scope of the currently considered linear reasoning. However, in view of the results given in the next section it can be stated in advance that, as said earlier, omitting considerations such as those according to Lemma 28 and Lemma 29 is justifiable in the present case. This is because, at a second glance, it will turn out that the adjoint disturbing potential fulfills Laplace's equation under certain approximations introduced in the section hereafter.

As always, after having acquired the desired linear field equation for the adjoint disturbing potential, the corresponding boundary condition has to be determined next. Starting from the boundary condition (6.85) given in the framework of the nonlinear problem, i.e.

$$\left(\mathbf{q}^\top \nabla_\xi \psi - \psi\right)\big|_\Sigma = v\,, \tag{6.98}$$

the following relation

$$\left(\mathbf{q}^\top \nabla_\xi \bar{\psi}_0 - \bar{\psi}_0\right)\big|_\Sigma = \bar{v}_0\,, \tag{6.99}$$

is obtained by using the adjoint normal potential $\bar{\psi}_0$, instead of the true adjoint potential ψ, in (6.83). Similarly to (5.95) and (5.96), the new ellipsoidal reference boundary values $\bar{v}_0$ in (6.99) refer likewise to the auxiliary space boundary surface Σ, in this case specified according to Definition 28. That is,

$$\bar{v}_0 = V_0^{ell}\big|_\Sigma \tag{6.100}$$

applies. The reason is again the identical mapping property assumed for the ellipsoidal regular gravity space transformation, which maps the surface Σ originally defined in the auxiliary space onto the same surface in ordinary space. Moreover, as is known, it appears that in (6.98) and (6.99), the vector $\boldsymbol{\xi}$ is restricted to the same boundary surface Σ. Hence, since the vector $\mathbf{q}$ is only dependent on $\boldsymbol{\xi}$, cf. (6.69), and since $\boldsymbol{\xi}$ is apparently the same vector in both equations given above, $\mathbf{q}$ is also one and the same vector. Thus, (6.98) and (6.99) are linear in ψ and $\bar{\psi}_0$ and it holds

$$\left(\mathbf{q}^\top \nabla_\xi \delta\bar{\psi} - \delta\bar{\psi}\right)\big|_\Sigma = \Delta\bar{v} = v - V_0^{ell}\big|_\Sigma\,. \tag{6.101}$$

This sets the stage for introducing the following BVP:

Definition 30 *The linear geodetic boundary value problem in ellipsoidal regular gravity space poses the following problem: the data, i.e. gravitational potential anomalies, are now given at the known surface* Σ

$$\Delta\bar{v} := \left(v - V_0^{ell}\big|_\Sigma\right) \;:\; \Sigma \to \mathbb{R}$$

and needed is a real function $\delta\bar{\psi}(\boldsymbol{\xi}) : ext\ \Sigma \to \mathbb{R}$

$$\begin{aligned} \mathrm{tr}\,\bar{\boldsymbol{\Phi}}_0 \mathrm{tr}\,\delta\bar{\boldsymbol{\Phi}} - \mathrm{tr}\,[\bar{\boldsymbol{\Phi}}_0 \delta\bar{\boldsymbol{\Phi}}] &= \frac{1}{2}\left(\mathrm{tr}\,\bar{\boldsymbol{\Phi}}_0^2 - \left(\mathrm{tr}\,\bar{\boldsymbol{\Phi}}_0\right)^2\right)\ , \quad \boldsymbol{\xi} \in ext\ \Sigma &(6.102)\\ \left(\mathbf{q}^\top \nabla_\xi \delta\bar{\psi} - \delta\bar{\psi}\right)\big|_\Sigma &= \Delta\bar{v}\,, &(6.103)\end{aligned}$$

which is the solution of a linear but inhomogeneous partial differential equation of second-order, (6.102), under the boundary condition (6.103).

In contrast to the linear BVP identified in Definition 23, which is based on the homogeneous Laplace's equation, the underlying field equation in Definition 30 is an inhomogeneous differential equation. Again, by means of (6.102) and (6.103), a BVP with a *fixed* boundary subject to a linear oblique boundary condition is established, since in contrast to the Earth's surface σ, the boundary Σ is a known surface, see (6.38).

In the end, the solution obtained for $\delta\bar{\psi}$ from the above problem is required at last to determine the surface of the Earth σ defined by the position vector $\mathbf{x}\big|_\sigma$. Similarly to (5.103), starting from (6.86) and taking (6.90) into account yields

$$\begin{aligned} \mathbf{x}\big|_\sigma = \bar{\gamma}(\boldsymbol{\xi}\big|_\Sigma)\boldsymbol{\pi}(\boldsymbol{\xi}\big|_\Sigma) = \bar{\gamma}(\boldsymbol{\xi}\big|_\Sigma)\nabla_\xi \psi(\boldsymbol{\xi}\big|_\Sigma) &= \bar{\gamma}(\boldsymbol{\xi}\big|_\Sigma)\nabla_\xi\left(\bar{\psi}_0(\boldsymbol{\xi}\big|_\Sigma) + \delta\bar{\psi}(\boldsymbol{\xi}\big|_\Sigma)\right)\\ &= \bar{\gamma}(\boldsymbol{\xi}\big|_\Sigma)\nabla_\xi \bar{\psi}_0(\boldsymbol{\xi}\big|_\Sigma) + \bar{\gamma}(\boldsymbol{\xi}\big|_\Sigma)\nabla_\xi \delta\bar{\psi}(\boldsymbol{\xi}\big|_\Sigma)\,. \qquad (6.104)\end{aligned}$$

As before in (5.104), it can be shown by means of laborious computations or by means of utilization of a computer algebra system, such as e.g. *Mathematica*, that

$$\left(\bar{\gamma}\nabla_\xi \bar{\psi}_0\right)\big|_\Sigma = \boldsymbol{\xi}\big|_\Sigma \tag{6.105}$$

is as a matter of fact true in the case under investigation. See Appendix B-2 for the detailed derivation. Again,

$$\bar{\boldsymbol{\zeta}} = \bar{\boldsymbol{\gamma}}(\boldsymbol{\xi}\big|_{\Sigma})\nabla_{\xi}\delta\bar{\psi}(\boldsymbol{\xi}\big|_{\Sigma}) \tag{6.106}$$

denotes the position correction or position anomaly vector, cf. (5.106). Consequently, the position vector $\mathbf{x}\big|_{\sigma}$ specifying the surface of the Earth results as

$$\mathbf{x}\big|_{\sigma} = \mathbf{x}\big|_{\Sigma} + \bar{\boldsymbol{\zeta}}\,, \tag{6.107}$$

i.e. the sum of the vector $\mathbf{x}\big|_{\Sigma}$, which is establishing the approximation surface Σ in ordinary space, see also Section 5-6, and of the position anomaly vector $\bar{\boldsymbol{\zeta}}$, which closes the gap between the approximation surface Σ and the Earth's surface σ, cf. Fig. 3.1.

Remark 30 In view of (6.106), (6.107) and in comparison to (5.105), (5.106) it is worthwhile to comment on the difference between the norms of the two anomaly vectors $\boldsymbol{\zeta}$ and $\bar{\boldsymbol{\zeta}}$. In the present ellipsoidal case, the vector $\mathbf{x}\big|_{\Sigma}$ is designed to describe a far better approximation of the true Earth's surface σ than in the spherical case discussed in the last chapter, in particular, at the end of Section 5-3. This gives rise to the conclusion that $\bar{\boldsymbol{\zeta}}$ is, compared to $\boldsymbol{\zeta}$, of considerably smaller magnitude, which will prove to be of consequence for the numerical realization of each problem. In fact, the importance of this finding with respect to the order of magnitude of the boundary values and for the analytical data continuation is highlighted in the next chapter.

6-4 Spherical approximation of the GBVP in ellipsoidal regular gravity space

In line with the considerations on the Molodensky's problem, Chapter 3, or on the regular gravity space approach, Chapter 5, *spherical approximation* has been a potent means to significantly simplify the underlying linearized problem at hand. Therefore, this section aims at performing the spherical approximation step in the framework of the ellipsoidal regular gravity space theory, in order to achieve a homogeneous problem of Laplace type, see also Remark 29. In more detail, spherical approximation, i.e. omitting terms of order $\mathrm{O}(J_2)$ and higher, on the level of the coefficients $\bar{\alpha}_{ik}, \bar{\gamma}_{ik}$ and $\bar{\beta}_{imk}$ and on the level of the functional matrix $\bar{\boldsymbol{\Phi}}_0$ will be accomplished next in four consecutive lemmata. Thereafter, a first theorem demonstrates that the adjoint normal potential fulfills a well-established partial differential equation if spherical approximation applies. A second theorem provides the field equation for the adjoint disturbing potenial. Hence, together with considerations on the underlying boundary condition, the first definition of the spherically approximated GBVP in ellipsoidal regular gravity space is attained. In addition, introduction of spherical coordinates leads to a second definition, which finally provides a representation of the GBVP in spherical approximation and in spherical coordinates. As a result, it can be forestalled that the adjoint disturbing potential in spherical approximation actually satisfies Laplace's equation. Moreover, a familiar expression is obtained for the corresponding boundary condition.

To begin with, a first Lemma provides the matrix $\bar{\boldsymbol{\gamma}} = [\bar{\gamma}_{ik}]$ in spherical approximation:

Lemma 39 *It holds*

$$\bar{\gamma}_{ik} = \gamma_{ik} + \mathrm{O}(J_2)\ . \tag{6.108}$$

Proof. According to Lemma 32,

$$\boldsymbol{\xi} = \boldsymbol{\xi}_0(\mathbf{p}) + \mathrm{O}(J_2) = -\sqrt{GM}\frac{\mathbf{p}}{\|\mathbf{p}\|^{3/2}} + \mathrm{O}(J_2) \tag{6.109}$$

is true. Hence, for the corresponding magnitude $\|\boldsymbol{\xi}\|$ follows directly

$$\|\boldsymbol{\xi}\| = \frac{\sqrt{GM}}{\|\mathbf{p}\|^{1/2}} + \mathrm{O}(J_2)\ . \tag{6.110}$$

Next, solving (6.109) for $\mathbf{p}$ at the same time determining $\|\mathbf{p}\|$ from (6.110), leads to

$$\mathbf{p} = -GM\frac{\boldsymbol{\xi}}{\|\boldsymbol{\xi}\|^3} + \mathrm{O}(J_2)\ . \tag{6.111}$$

Note, that this result is obviously in accordance with (5.33). After this, the following derivations are based on Lemma 35. At first, taking (6.109) into consideration and then applying the quotient rule of differential calculus, results in

$$\begin{aligned}
[\bar{\gamma}_{ik}] &= \left[\frac{\partial \xi_i}{\partial p_k}\right] \\
&= \frac{\partial}{\partial p_k}\left(-\sqrt{GM}\frac{\mathbf{p}}{\|\mathbf{p}\|^{3/2}}\right) + \mathrm{O}(J_2) \\
&= \frac{-\sqrt{GM}}{\|\mathbf{p}\|^3}\left[\delta_{ik}\|\mathbf{p}\|^{3/2} - \frac{3}{2}\|\mathbf{p}\|^{-1/2} p_i p_k\right] + \mathrm{O}(J_2) \\
&= -\frac{\sqrt{GM}}{\|\mathbf{p}\|^{3/2}}\left[\delta_{ik} - \frac{3}{2}\frac{p_i p_k}{\|\mathbf{p}\|^2}\right] + \mathrm{O}(J_2)\ .
\end{aligned} \tag{6.112}$$

Thus, using (6.111) to rewrite (6.112) and by comparing the result to Lemma 21 yields

$$\begin{aligned}
[\bar{\gamma}_{ik}] &= -\frac{\|\boldsymbol{\xi}\|^3}{GM}\left[\delta_{ik} - \frac{3}{2}\frac{\xi_i \xi_k}{\|\boldsymbol{\xi}\|^2}\right] + \mathrm{O}(J_2) \\
&= [\gamma_{ik}] + \mathrm{O}(J_2)\ . \qquad \diamondsuit
\end{aligned}$$

Next, a second lemma addresses the task of accomplishing spherical approximation for the inverse matrix $\bar{\boldsymbol{\alpha}} = [\bar{\alpha}_{ik}]$:

Lemma 40 *It holds*

$$\bar{\alpha}_{ik} = \alpha_{ik} + \mathrm{O}(J_2)\ . \tag{6.113}$$

Proof. By definition, cp. Lemma 34 and Lemma 35 or (6.62), the matrix $[\bar{\alpha}_{ik}]$ results as the inverse of $[\bar{\gamma}_{ik}]$. Additionally, making use of (6.108) and comparing the result to Lemma 20 leads to

$$\begin{aligned}
[\bar{\alpha}_{ik}] &= [\bar{\gamma}_{ik}]^{-1} \\
&= \left[-\frac{\|\boldsymbol{\xi}\|^3}{GM}\left[\delta_{ik} - \frac{3}{2}\frac{\xi_i \xi_k}{\|\boldsymbol{\xi}\|^2}\right] + \mathrm{O}(J_2)\right]^{-1} \\
&= \left[-\frac{GM}{\|\boldsymbol{\xi}\|^3}\left[\delta_{ik} - 3\frac{\xi_i \xi_k}{\|\boldsymbol{\xi}\|^2}\right]\right] + \mathrm{O}(J_2) \\
&= [\alpha_{ik}] + \mathrm{O}(J_2)\ . \qquad \diamondsuit
\end{aligned}$$

Furthermore, a third lemma is devoted to install a spherically approximated version for the function $\bar{\beta}_{imk}$:

Lemma 41 *It holds*

$$\bar{\beta}_{imk} = \beta_{imk} + \mathrm{O}(J_2)\ . \tag{6.114}$$

Proof. Starting from Lemma 36, thereby accounting for (6.108) again, and taking Lemma 24 into consideration, produces

$$\begin{aligned}
\bar{\beta}_{imk} &= \frac{\partial \bar{\gamma}_{im}}{\partial p_k} \\
&= \frac{\partial \gamma_{im}}{\partial p_k} + \mathrm{O}(J_2) \\
&= \beta_{imk} + \mathrm{O}(J_2)\ . \qquad \diamondsuit
\end{aligned}$$

A fourth and last lemma sets the stage for simplifying the underlying field equation of the BVP established in Definition 30 by introducing a spherical approximation variant for the functional matrix $\bar{\boldsymbol{\Phi}}_0 = \left[\bar{\Phi}^0_{ij}\right]$:

Lemma 42 *It holds*

$$\bar{\Phi}^0_{ij} = \gamma_{ij} + \mathrm{O}(J_2)\ . \tag{6.115}$$

Proof. In analogy to the proof given in conjunction with Lemma 28, it is advantageous to achieve spherical approximation for the matrix $\bar{\boldsymbol{\Phi}}_0 = \left[\bar{\Phi}^0_{ij}\right]$, (6.91), in a three step procedure. At first, $\bar{\psi}_0$, (6.88), is expressed by means of ψ_0, (5.72), in the following form

$$\bar{\psi}_0 = \psi_0 + \mathrm{O}(J_2)\ , \tag{6.116}$$

then, in view of (5.76), the following reasoning obviously holds true

$$\frac{\partial^2 \bar{\psi}_0}{\partial \xi_m \partial \xi_n} = \frac{\partial^2 \psi_0}{\partial \xi_m \partial \xi_n} + \mathrm{O}(J_2) = \frac{\partial^2 \left(-\frac{2GM}{\|\boldsymbol{\xi}\|}\right)}{\partial \xi_m \partial \xi_n} + \mathrm{O}(J_2) = -2\alpha_{mn} + \mathrm{O}(J_2)\ . \tag{6.117}$$

Hence, due to Lemma 39 and with (5.77) given,

$$\bar{\gamma}_{im}\bar{\gamma}_{jn}\frac{\partial^2 \bar{\psi}_0}{\partial \xi_m \partial \xi_n} = \gamma_{im}\gamma_{jn}\frac{\partial^2 \psi_0}{\partial \xi_m \partial \xi_n} + \mathrm{O}(J_2) = \gamma_{im}\gamma_{jn}\frac{\partial^2 \left(-\frac{2GM}{\|\boldsymbol{\xi}\|}\right)}{\partial \xi_m \partial \xi_n} + \mathrm{O}(J_2) = -2\gamma_{ij} + \mathrm{O}(J_2) \tag{6.118}$$

follows further on. Next, by means of Lemma 41 and by taking (5.78) into account,

$$\bar{\beta}_{imj}\frac{\partial \bar{\psi}_0}{\partial \xi_m} = \beta_{imj}\frac{\partial \psi_0}{\partial \xi_m} + \mathrm{O}(J_2) = \beta_{imj}\frac{\partial \left(-\frac{2GM}{\|\boldsymbol{\xi}\|}\right)}{\partial \xi_m} + \mathrm{O}(J_2)\ = 3\gamma_{ij} + \mathrm{O}(J_2) \tag{6.119}$$

is derived. Finally, using (6.118) and (6.119), yields $\bar{\boldsymbol{\Phi}}_0 = \left[\bar{\Phi}^0_{ij}\right]$ in spherical approximation

$$\begin{aligned}
\left[\bar{\Phi}^0_{ij}\right] &= \left[\bar{\gamma}_{im}\bar{\gamma}_{jn}\frac{\partial^2 \bar{\psi}_0}{\partial \xi_m \partial \xi_n} + \bar{\beta}_{imj}\frac{\partial \bar{\psi}_0}{\partial \xi_m}\right] \\
&= \left[\gamma_{im}\gamma_{jn}\frac{\partial^2 \left(-\frac{2GM}{\|\boldsymbol{\xi}\|}\right)}{\partial \xi_m \partial \xi_n} + \beta_{imj}\frac{\partial \left(-\frac{2GM}{\|\boldsymbol{\xi}\|}\right)}{\partial \xi_m}\right] + \mathrm{O}(J_2) \\
&= \left[\gamma_{ij}\right] + \mathrm{O}(J_2)\ . \qquad \diamondsuit
\end{aligned}$$

First of all, this provides the basis for simplifying the right-hand side of (6.94) as demonstrated in the next theorem:

Theorem 13 *In spherical approximation the adjoint normal potential $\bar{\psi}_0$, (6.88), fulfills the following partial differential equation*

$$\left(\mathrm{tr}\,\bar{\boldsymbol{\Phi}}_0\right)^2 - \mathrm{tr}\,\bar{\boldsymbol{\Phi}}_0^2 = 0\,. \tag{6.120}$$

Proof. Having regard to Lemma 42 as well as to Lemma 29 thereafter, (6.120) is verified as follows

$$\begin{aligned}
\left(\mathrm{tr}\,\bar{\boldsymbol{\Phi}}_0\right)^2 - \mathrm{tr}\,\bar{\boldsymbol{\Phi}}_0^2 &= \left(\mathrm{tr}\,\boldsymbol{\gamma}\right)^2 - \mathrm{tr}\,\boldsymbol{\gamma}^2 + \mathrm{O}(J_2) \\
&= \frac{9}{4}\frac{\|\boldsymbol{\xi}\|^6}{GM^2} - \frac{\|\boldsymbol{\xi}\|^6}{GM^2}\mathrm{tr}\left[\delta_{ij} - \frac{3}{4}\frac{\xi_i\xi_j}{\|\boldsymbol{\xi}\|^2}\right] + \mathrm{O}(J_2) \\
&= \frac{\|\boldsymbol{\xi}\|^6}{GM^2}\left(\frac{9}{4} - \frac{9}{4}\right) + \mathrm{O}(J_2) \\
&= 0 + \mathrm{O}(J_2)\ . \qquad \diamondsuit
\end{aligned}$$

This answers the question that came up when giving the proof for Theorem 12. Thus, it could be evidenced that $\left(\mathrm{tr}\,\bar{\boldsymbol{\Phi}}_0\right)^2 - \mathrm{tr}\,\bar{\boldsymbol{\Phi}}_0^2$ is indeed small of order $\mathrm{O}(J_2)$, see also Remark 29. Above all, by dint of this finding and the results presented in the previous lemma, the inhomogeneous field equation (6.94) can at last be replaced by the homogeneous Laplace's equation if terms of $\mathrm{O}(J_2)$ and higher are disregarded henceforth:

Theorem 14 *In spherical approximation the linearized field equation constituted in Theorem 12 becomes*

$$\Delta\delta\bar{\psi} = 0\,. \tag{6.121}$$

Proof. The line of argument is based on the principle to reduce the ellipsoidal field equation back to the spherical case. The latter has been elaborated in detail within Theorem 8 and is only shortly reviewed here

$$\begin{aligned}
0 &= \mathrm{tr}\left[\bar{\Phi}^0_{ij}\right]\mathrm{tr}\left[\delta\bar{\Phi}_{ij}\right] - \mathrm{tr}\left[\left[\bar{\Phi}^0_{ij}\right]\left[\delta\bar{\Phi}_{jk}\right]\right] \\
&= \mathrm{tr}\left[\gamma_{ij}\right]\mathrm{tr}\left[\gamma_{im}\gamma_{jn}\frac{\partial^2\delta\bar{\psi}}{\partial \xi_m \partial \xi_n} + \beta_{imj}\frac{\partial\delta\bar{\psi}}{\partial \xi_m}\right] - \mathrm{tr}\left[\left[\gamma_{ij}\right]\left[\gamma_{jm}\gamma_{kn}\frac{\partial^2\delta\bar{\psi}}{\partial \xi_m \partial \xi_n} + \beta_{jmk}\frac{\partial\delta\bar{\psi}}{\partial \xi_m}\right]\right] \\
&= -\frac{3}{2}\frac{\|\boldsymbol{\xi}\|^3}{GM}\mathrm{tr}\left[\frac{\|\boldsymbol{\xi}\|^6}{GM^2}\left[\delta_{im} - \frac{3}{2}\frac{\xi_i\xi_m}{\|\boldsymbol{\xi}\|^2}\right]\left[\delta_{jn} - \frac{3}{2}\frac{\xi_j\xi_n}{\|\boldsymbol{\xi}\|^2}\right]\frac{\partial^2\delta\bar{\psi}}{\partial \xi_m \partial \xi_n} - \right. \\
&\qquad \left. \frac{3}{2}\frac{\|\boldsymbol{\xi}\|^4}{GM^2}\left(\delta_{im}\xi_j + \delta_{mj}\xi_i + \delta_{ij}\xi_m - \frac{7}{2}\frac{\xi_i\xi_m\xi_j}{\|\boldsymbol{\xi}\|^2}\right)\frac{\partial\delta\bar{\psi}}{\partial \xi_m}\right]
\end{aligned}$$

$$
\begin{aligned}
&-\mathrm{tr}\left[\frac{-\|\boldsymbol{\xi}\|^9}{GM^3}\left[\delta_{im}-\frac{3}{4}\frac{\xi_i\xi_m}{\|\boldsymbol{\xi}\|^2}\right]\left[\delta_{kn}-\frac{3}{2}\frac{\xi_k\xi_n}{\|\boldsymbol{\xi}\|^2}\right]\frac{\partial^2\delta\bar{\psi}}{\partial\xi_m\partial\xi_n}+\right.\\
&\qquad\left.\frac{3}{2}\frac{\|\boldsymbol{\xi}\|^7}{GM^3}\left(\delta_{im}\xi_k-\frac{1}{2}\delta_{mk}\xi_i+\delta_{ik}\xi_m-\frac{5}{4}\frac{\xi_i\xi_k\xi_m}{\|\boldsymbol{\xi}\|^2}\right)\frac{\partial\delta\bar{\psi}}{\partial\xi_m}\right]\\
=&-\frac{3}{2}\frac{\|\boldsymbol{\xi}\|^9}{GM^3}\left(\delta_{mn}-\frac{3}{4}\frac{\xi_m\xi_n}{\|\boldsymbol{\xi}\|^2}\right)\frac{\partial^2\delta\bar{\psi}}{\partial\xi_m\partial\xi_n}+\frac{27}{8}\frac{\|\boldsymbol{\xi}\|^7}{GM^3}\xi_m\frac{\partial\delta\bar{\psi}}{\partial\xi_m}+\\
&\quad\frac{\|\boldsymbol{\xi}\|^9}{GM^3}\left(\delta_{mn}-\frac{9}{8}\frac{\xi_m\xi_n}{\|\boldsymbol{\xi}\|^2}\right)\frac{\partial^2\delta\bar{\psi}}{\partial\xi_m\partial\xi_n}-\frac{27}{8}\frac{\|\boldsymbol{\xi}\|^7}{GM^3}\xi_m\frac{\partial\delta\bar{\psi}}{\partial\xi_m}\\
=&-\frac{1}{2}\frac{\|\boldsymbol{\xi}\|^9}{GM^3}\delta_{mn}\frac{\partial^2\delta\bar{\psi}}{\partial\xi_m\partial\xi_n}\\
=&\ \Delta\delta\bar{\psi}\,. \qquad \diamondsuit
\end{aligned}
$$

Hence, by means of spherical approximation a satisfying result has been achieved. That is to say, the corresponding field equation for the adjoint disturbing potential $\delta\bar{\psi}$ is found to be Laplace's equation. Again, it remains to elucidate the question about the associated boundary condition. For that purpose, (6.103), i.e.

$$\left(\mathbf{q}^\top\nabla_\xi\delta\bar{\psi}-\delta\bar{\psi}\right)\big|_\Sigma=\Delta\bar{v}\,, \tag{6.122}$$

the boundary condition of the linear GBVP in ellipsoidal regular gravity space as derived in the last section is recalled. In this regard, it is important that the *Mathematica* expression acquired for vector $\mathbf{q}$, cf. Appendix B-1, is of the following form

$$\mathbf{q}(\boldsymbol{\xi})=\mathbf{q}_0(\boldsymbol{\xi})+J_2\mathbf{q}_1(\boldsymbol{\xi})+J_2^2\mathbf{q}_2(\boldsymbol{\xi})\,. \tag{6.123}$$

Thus, in terms of spherical approximation, that is for $J_2=0$ in (B.1), results

$$\mathbf{q}(\boldsymbol{\xi})=\mathbf{q}_0(\boldsymbol{\xi})=-\frac{1}{2}\boldsymbol{\xi}\,. \tag{6.124}$$

Consequently, on the basis of (6.122), thereby accounting for the last result given in (6.124), and together with (6.121), the following BVP can be introduced:

Definition 31 *In ellipsoidal regular gravity space, the geodetic boundary value problem in spherical approximation reflects the following problem: the data, i.e. gravitational potential anomalies, are given at the known surface* Σ

$$\Delta\bar{v}:=\left(v-V_0^{ell}\big|_\Sigma\right)\ :\ \Sigma\to\mathbb{R}$$

and one must find a real function $\delta\bar{\psi}(\boldsymbol{\xi}):ext\ \Sigma\to\mathbb{R}$

$$\Delta\delta\bar{\psi}(\boldsymbol{\xi}) = 0\,,\quad \boldsymbol{\xi}\in ext\ \Sigma \tag{6.125}$$

$$-\left(\frac{1}{2}\boldsymbol{\xi}^\top\nabla_\xi\delta\bar{\psi}+\delta\bar{\psi}\right)\Bigg|_\Sigma = \Delta\bar{v}\,, \tag{6.126}$$

which is now the solution of a linear and homogeneous partial differential equation of second-order, (6.125), under the boundary condition (6.126).

It is quite obvious that by means of spherical approximation the designated simplification of the linear BVP constituted by Definition 30 has been accomplished. More precisely, instead of the necessity to fulfill the inhomogeneous partial differential equation (6.102), the adjoint disturbing potential results as the solution of Laplace's equation. Furthermore, the above boundary condition is reobtained in the well-established form according to (5.101). Altogether, the problem specified by Definition 31 is a linear fixed BVP with an oblique boundary condition.

As seen before in Section 3-6 and Section 5-4, introduction of spherical coordinates proved to be beneficial in order to simplify the underlying boundary conditions. Hence, the aim is to take advantage of a spherical coordinates representation to also revise (6.126). For this purpose, the position vector $\boldsymbol{\xi}$ in ellipsoidal regular gravity space is expressed in terms of spherical coordinates (λ,ϕ,r), cf. (2.1), as follows

$$\boldsymbol{\xi}=r_\xi\left[\begin{array}{ccc}\cos\lambda_\xi\cos\phi_\xi & \sin\lambda_\xi\cos\phi_\xi & \sin\phi_\xi\end{array}\right]^\top. \tag{6.127}$$

In a similar manner as previously applied in Section 5-4, the gradient of the adjoint disturbing potential $\nabla_\xi\delta\bar{\psi}$ in (6.126) is simply replaced by the radial derivative $\frac{\partial}{\partial r_\xi}$. Again, this seems practicable since the adjoint disturbing potential $\delta\bar{\psi}$ is a small quantity, which, in addition, is dominated by radial influence. In fact, this proceeding can be even better motivated in the ellipsoidal case under investigation due to the fact that the magnitude of the adjoint disturbing potential $\delta\bar{\psi}$ should scale down compared to the magnitude of the adjoint disturbing potential $\delta\psi$ involved in the framework of the approach presented in the last chapter. As a result, the committed error is expected to scale down in the same way. Thus, with Definition 24 in mind, the following BVP can be characterized:

Definition 32 *In ellipsoidal regular gravity space, the geodetic boundary value problem in spherical approximation and based on a representation in spherical coordinates reflects the following problem: the data, i.e. gravitational potential anomalies, are given at the known surface* Σ

$$\Delta\bar{v} := \left(v - V_0^{ell}\big|_{\Sigma}\right) \ : \ \Sigma \to \mathbb{R}$$

and to be found is a real function $\delta\bar{\psi}(\boldsymbol{\xi}) : ext\ \Sigma \to \mathbb{R}$

$$\Delta\delta\bar{\psi}(\boldsymbol{\xi}) = 0\,, \quad \boldsymbol{\xi} \in ext\ \Sigma \tag{6.128}$$

$$-\left(\frac{1}{2} r_\xi \frac{\partial\delta\bar{\psi}}{\partial r_\xi} + \delta\bar{\psi}\right)\Bigg|_{\Sigma} = \Delta\bar{v}\,, \tag{6.129}$$

which is the solution of Laplace's partial differential equation, (6.125), under the boundary linear oblique condition (6.126).

Even though the overall problem remains the same, that is (6.128) and (6.129) establish a linear fixed BVP based on an oblique boundary condition, the achievement to be pointed out with regard to Definition 31 and Definition 32 is that the inner product of $\boldsymbol{\xi}$ and $\nabla_\xi \delta\bar{\psi}$ gives way to an ordinary multiplication of the two scalar-valued quantities r_ξ and $\frac{\partial\delta\bar{\psi}}{\partial r_\xi}$. Moreover, focusing on a radial consideration offers the possibility to conduct constant radius approximation as envisaged in the next section.

At last, by means of $\delta\bar{\psi}$ deduced as the solution from either of the two BVPs given above, the Earth's surface σ is derived again according to (6.106) and (6.107).

Remark 31 At this point it is advisable to recapitulate for the different approaches presented so far what have been the particular assumptions with regard to the respective *spherical approximation* step. E.g., within the scope of the classical treatment of Molodensky's problem, a spherical reference potential to simplify the boundary condition together with a spherical coordinate representation has been applied to obtain the simple Molodensky's problem. Whereas in the context of the regular gravity space approach, Chapter 5, spherical approximation was closely connected to the application of spherical coordinates and a restriction to the radial derivatives. Note that a spherical reference potential was already introduced in the prior linearization step. At last, spherical approximation in the present case has to be understood primarily in the sense of assuming $J_2 = 0$ for all involved quantities. That is, for the coefficients $\bar{\alpha}_{ij}$, $\bar{\beta}_{imj}$ and $\bar{\gamma}_{ij}$ as well as for the adjoint normal potential $\bar{\psi}_0$. Beyond it, the representation in spherical coordinates, again only accounting for the radial derivatives, is additionally implied. However, in contrast to the first two approaches, spherical approximation within the scope of the ellipsoidal approach is not only utilized to simplify the underlying boundary condition, cf. (6.126) and (6.129) respectively, but also to approximate the corresponding field equation, see Theorem 13 and Theorem 14.

6-5 Constant radius approximation of the GBVP in ellipsoidal regular gravity space

The following section is meant to provide, similarly to Section 5-5 and in view of the numerical studies intended for in the next chapter, a representation for the GBVP in ellipsoidal regular gravity space particularly suitable for the subsequent computational realization. To this end, the special case of identical mapping, cf. Lemma 33, is first recalled. That is, from Lemma 33 follows that the independent coordinate vector $\boldsymbol{\xi}$ in auxiliary space equals the independent coordinate vector $\mathbf{x}$ in ordinary space if the true potential V reduces to the ellipsoidal normal scenario described by V_0^{ell}. On the other hand, giving up this ellipsoidal restriction by adopting a more realistic function for V leads to a perturbation of the identity (6.43). However, it can still be claimed that approximately

$$\boldsymbol{\xi} \approx \mathbf{x} \tag{6.130}$$

holds. Now, in consideration of (6.130) and by taking up the same argumentation as has already been given in the framework of Section 5-5, certain approximations on the level of the underlying spherical coordinate representations for $\mathbf{x}$ and $\boldsymbol{\xi}$ are feasible. More precisely, starting from the position vector $\mathbf{x}$ given in spherical coordinates (λ, ϕ, r) as follows

$$\mathbf{x} = r \left[\begin{array}{ccc} \cos\lambda\cos\phi & \sin\lambda\cos\phi & \sin\phi \end{array} \right]^\top ,$$

the associated position vector $\boldsymbol{\xi}$ in regular gravity space

$$\boldsymbol{\xi} = r_\xi \left[\begin{array}{ccc} \cos\lambda_\xi\cos\phi_\xi & \sin\lambda_\xi\cos\phi_\xi & \sin\phi_\xi \end{array} \right]^\top$$

transforms by neglecting in the previous two expressions the longitudinal and latitudinal differences, i.e.

$$\lambda_\xi = \lambda \quad ; \quad \phi_\xi = \phi \,, \tag{6.131}$$

and, in addition, by adopting

$$r_\xi = (R + h_\xi) \tag{6.132}$$

into the following relationship

$$\boldsymbol{\xi} = (R + h_\xi) \left[\begin{array}{ccc} \cos\lambda\cos\phi & \sin\lambda\cos\phi & \sin\phi \end{array} \right]^\top . \tag{6.133}$$

As already discussed before in Section 5-5, (6.133) is of practical importance, since due to (6.131) corresponding points in ordinary and auxiliary space are located virtually in the same radial direction. This will be advantageous for the numerical proof of concept given right after this chapter.

Moreover, by setting

$$h_\xi = 0 \tag{6.134}$$

in (6.133), thus by again accepting an error of h_ξ/R, the following BVP can be defined:

Definition 33 *In ellipsoidal regular gravity space, the geodetic boundary value problem in constant radius approximation reflects the following problem: the data, i.e. gravitational potential anomalies, are now given at the reference sphere S*

$$\Delta\bar{v} := \left(v - V_0^{ell}\big|_\Sigma\right) \;:\; S \to \mathbb{R}$$

and required is a real function $\delta\bar{\psi}(\boldsymbol{\xi}) : ext\ S \to \mathbb{R}$

$$\Delta\delta\bar{\psi}(\boldsymbol{\xi}) \;=\; 0\,, \quad \boldsymbol{\xi} \in ext\ S \tag{6.135}$$

$$-\left(\frac{1}{2} r_\xi \frac{\partial\delta\bar{\psi}}{\partial r_\xi} + \delta\bar{\psi}\right)\bigg|_S \;=\; \Delta\bar{v}\,, \tag{6.136}$$

which is the solution of Laplace's partial differential equation, (6.125), under the linear boundary condition (6.126).

As usual, see also Section 3-7 and Section 5-5, the above BVP results by mapping the boundary data $\Delta\bar{v}$ from the boundary surface Σ to the corresponding reference sphere S. As is known, the sphere S is, with regard to the later computational part, the preferred mathematical reference surface. In addition, the BVP given in Definition 33 represents a fixed and, beyond that, a normal derivative problem since in (6.136) the direction $\frac{\partial}{\partial r_\xi}$ coincides with the normal to the sphere S. This is in contrast to the former problem introduced in Definition 32, which is based on the oblique boundary condition (6.129).

Determination of the Earth's surface σ is achieved again after having solved the above BVP for $\delta\bar{\psi}$. For that purpose, the utilization of the sphere S, as is the case in Definition 33, is of no account. As discussed already in Section 5-5, the true approximation surface for σ remains Σ. The sphere S serves only as a computational reference surface. Hence, the Earth's surface is still obtained from (6.106) and (6.107).

Remark 32 Within the scope of Remark 10 and Remark 26 it has been pointed out that for accuracy reasons it might be advisable to consider the use of strategies to analytically continue the boundary data from the actual boundary surface Σ onto the sphere S. The latter is of importance when it gets down to the numerical experiment. However, it is anticipated in the present context that the necessity for analytical data continuation becomes possibly negligible. The reason for this expectation is the reduced separation between the Earth's surface σ and the boundary surface Σ. Due to the fact that Σ, besides constituting the boundary surface in auxiliary space, also serves as the reference surface for the normal potential data, see also (6.100), a reduced separation leads to higher quality normal data. Since in addition the normal potential V_0^{ell} is a more realistic approximation for the true potential V than V_0, altogether a considerable reduction in terms of magnitude of the boundary data $\bar{v}_0$ compared to v_0 follows out of it. As a result, in the framework of the ellipsoidal approach, a simple mapping of the boundary data ought to be sufficient. This unsettled aspect will be clarified after giving a short summary of this chapter.

6-6 Brief discussion on the benefit of an ellipsoidal concept

First of all, after completing all theoretical scrutinies on both regular gravity space approaches, nearly a total symmetry in terms of underlying mathematical formulae can be claimed between the regular gravity space approach presented in the last chapter and the ellipsoidal regular gravity space approach introduced in this chapter. Hence, all the conclusions given in Section 5-6 concerning the comparison between the gravity space approach according to F. Sansò and the first regular gravity space approach, Chapter 5, could be adopted for the same comparison between F. Sansò's approach and the regular approach at last presented. The reason is that essentially the two regular approaches differ only in the use of different linearization points, the overall formalism, as said earlier, remains the same. This can easily be confirmed since the BVPs established according to Definition 25 and Definition 33 are mathematically of the same structure. The application of different linearization points is solely reflected in the underlying boundary data, which, as a result, influences the acquired adjoint disturbing potential but not the mathematical form of the BVP.
However, besides the described analogousness, a few minor differences in the two regular approaches also have to be admitted. First of all, in consideration of the coordinate mapping between ordinary and auxiliary space it has to be pointed out that the closed-form transformation formulae, see Definition 20, are replaced by the series representation according to Definition 27. Naturally, a coordinate transformation based on a series expansion involves a certain level of approximation. The influence thereof has to be taken into account. Furthermore, attention must be called to the fact that in the framework of the ellipsoidal method, Laplace's equation has been established as the relevant field equation only after having conducted the spherical approximation step. This is in contrast to the gravity space approach introduced in Chapter 5, where Laplace's equation was already identified as the underlying field equation after the linearization step. Hence, within the scope of the approach based on the ellipsoidal normal concept a higher level of approximations or, in other words, more assumptions are required to obtain the same type of partial differential equation, which has to be satisfied by the adjoint disturbing potential. The appropriateness of these assumptions is investigated in the subsequent numerics chapter.
On the other hand, despite the addressed restrictions, the inherent advantages to the approach based on the ellipsoidal theory prevail. As discussed before, the ellipsoidal normal potential V_0^{ell} is a better approximation for the true potential V than the simple spherical normal potential V_0. Furthermore, the use of V_0^{ell} also involves a reference surface Σ for the normal boundary values, which better approximates the true Earth's surface σ as is the case if Σ results from V_0. Considered together, the boundary data associated with the ellipsoidal approach are expected to have a positive impact on the numerical realization of the BVP solution. Whether this is the case will turn out in the next chapter.

Finally, the table set out hereafter compares all four approaches given for the GBVP in this work in terms of the underlying basic principles and properties:

	GBVP approach according to			
author and date of origin	M.S. Molodensky 1945	F. Sansò 1976	W. Keller 1986	G. Austen, W. Keller 2006
Chapter	Ch. 3	Ch. 4	Ch. 5	Ch. 6
	BVP classification			
BVP category	vectorial free external regular	fixed internal singular at origin	fixed external regular	fixed external regular
space domain and associated name	ordinary geometry space	auxiliary gravity space	auxiliary regular grav. sp.	auxiliary ellip. reg. grav. sp.
	underlying transformation			
general type	—	Legendre transf.	contact transf.	contact transf.
w.r.t. coordinates	—	see (4.5), (4.15)	see (5.1), (5.41)	see (6.37), (6.59)
w.r.t potential	—	see (4.19), (4.20)	see (5.2), (5.18)	see (6.39), (6.68)
representation	—	closed-form	closed-form	series expansion
	relevant reference surfaces			
physical surface	Earth's surface σ	Earth's surface σ	Earth's surface σ	Earth's surface σ
in ordinary space	Marussi telluroid Σ see (3.4)	grav. telluroid Σ_g see (4.61)	—	—
in auxiliary space	—	boundary surface Σ see (4.6)	—	—
in ordinary and auxiliary space	—	—	grav. telluroid Σ see (5.14)	boundary surface Σ see (6.38)
	linearization point			
reference potential	W_0	ψ_0 see (4.38)	ψ_0 see (5.72)	$\bar{\psi}_0$ see (6.88)
reference surface	Marussi telluroid Σ	—	—	—
	corresponding boundary data in			
nonlinear case	w, $\tilde{\mathbf{\Gamma}}$ (see Def. 5)	v (see Def. 17)	v (see Def. 22)	v (see Def. 29)
linear case	Δw $\Delta\Gamma'$ (see Def. 10)	Δv (see Def. 19)	Δv (see Def. 23)	$\Delta\bar{v}$ (see Def. 30)
spherical approx.	$\Delta\Gamma$ (see Def. 11)	—	Δv (see Def. 24)	$\Delta\bar{v}$ (see Def. 32)

Table 6.1: Overview of the discussed GBVP approaches

Chapter 7

Numerical proof of concept

For the first time, a numerical proof of concept will be given for the regular gravity space approach, which has been derived by W. Keller in the 1980's, see Chapter 5. Furthermore, the improved feasibility of the refined regular approach, which is based on an ellipsoidal scenario as introduced in Chapter 6, will also be demonstrated by means of the following investigations. As far as the gravity space approach is concerned, which is based on the formulation pioneered by F. Sansò in the 1970's, it can be stated that the first numerical experiments utilizing this method have already been conducted in the work of [1] Antunes 2004. In brief, the focus of that contribution was on the problem of finding a methodology in gravity space for the practical geoid determination problem. The numerical realization took place in the framework of a regional case study based on real measurements obtained in a field campaign for a test area in central Portugal. As a consequence of such a local survey, instead of a spherical harmonics representation for the terrestrial potential, a point mass method has been employed to model the Earth's potential or the geoid. Due to the nature of the setup and the methods applied for that experiment, it was found that the proposed solution procedure for the geoid must be considered a relative determination method to successively improve an already existing geoid solution in an iterative procedure rather than an absolute determination method, which is independent form an a priori geoid solution. Anyhow, the general applicability of F. Sansò's gravity space approach for solving the GBVP and, in particular, for the geoid determination problem was successfully demonstrated by C. Antunes. However, the aim of this chapter is to verify first of all the appropriateness of the theoretical concepts of the two regular gravity space approaches. To attain a real solution based on actual observed data does not have top priority at this stage. In fact, a globally supported result with respect to an assessment of the suitability of the proposed approaches based on regular formulations is particularly desired.

For this purpose, see also [2] Austen 2007, a numerical closed-loop study for a global simulation scenario has been devised. The investigations on both regular gravity space approaches comply with the testing procedure outlined in the flowchart presented in Fig. 7.1. According to this processing scheme, within the framework of a global spherical harmonic synthesis (GSHS), the Earth's surface σ, the gravitational potential values $V\big|_\sigma$ and the corresponding gravitational acceleration vectors $\nabla V\big|_\sigma$ are generated at ground level. As a result, the GSHS step is based on known global models for the Earth's topography (GTM) and for the potential field of the Earth (GPM). Thereafter, based on the gradient vectors $\nabla V\big|_\sigma$ of the gravitational potential and by means of the underlying transformation methods of the respective gravity space approach (GSA), the boundary surface Σ of gravity space is synthesized. The corresponding boundary values on Σ, i.e. the potential anomalies $\Delta v\big|_\Sigma$, result from the ground potential values $\nabla V\big|_\sigma$ together with the normal potential data V_0 derived on the basis of a global reference model (GRM). With the subsequent analysis step in mind, the boundary data must be shifted from the actual boundary surface to another auxiliary reference surface required for the numerical evaluation. In general, a simple mathematical surface like the sphere or the ellipsoid of revolution is chosen. The boundary data is transfered from the original boundary surface to the new boundary surface via analytical data continuation (ADC). In the present case, the analysis step relies on the sphere S as a computational reference and is performed in terms of a global spherical harmonic analysis (GSHA). The aim of the GSHA step is to recover the field parameters (Stokes coefficients) of the adjoint disturbing potential $\delta\psi$. By means of $\delta\psi$ the position anomaly vector $\boldsymbol{\zeta}$ can be determined. The Earth's surface σ results from its approximation, more precisely from the surface Σ, together with the position anomaly vector $\boldsymbol{\zeta}$. A comparison of the recovered and the real surface of the Earth completes the closed-loop study and allows for evaluation of the suitability of the gravity space methods.

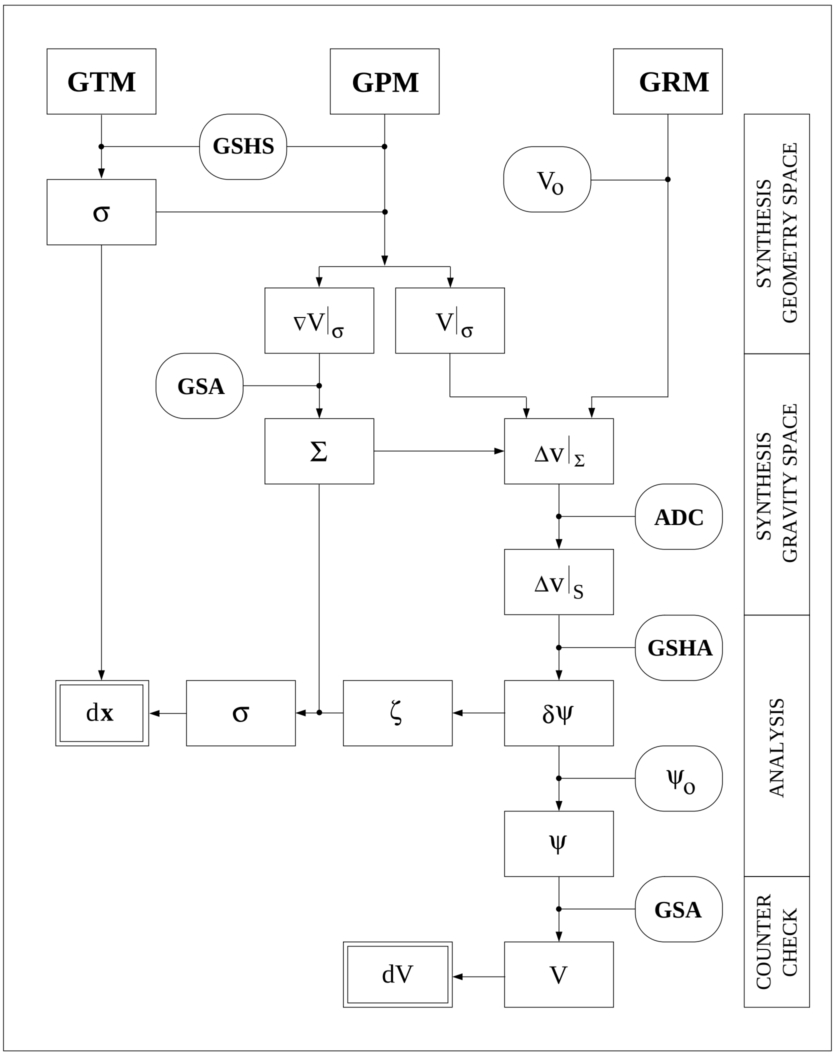


Figure 7.1: Flowchart of the proof of concept.

Besides assessing the capability of the proposed gravity space methodology on the geometrical level, a further verification on the potential level is feasible. For this purpose, the adjoint potential ψ is obtained from the previously derived adjoint disturbing potential $\delta\psi$ and from the known adjoint normal potential ψ_0. Thereafter, using the transformation formulae provided by the GSA, ψ is converted back into the initial potential V. A final comparison of the recovered and the true potential again gives information about the applicability of the gravity space approach for solving the GBVP. For simplicity and clearness reasons, the final error evaluation is conducted on the geometrical level. Of course, the evaluation on the potential level would lead to the same results.

The studies conducted in the next two sections follow the above outlined working plan. First of all, further considerations and numerical experiments will be given for the regular gravity space approach based on an isotropic linearization point, cf. Chapter 5. Accordingly, the regular gravity approach based on a spheroidal linearization point, see Chapter 6, will be numerically investigated in the section thereafter. It can already be stated that in the framework of the numerical simulations performed for both cases, the following input models will be used. For the global topography, the TUG87 model from M. Wieser, see [103] Wieser 1987, given in terms of the coefficients for a spherical harmonics series expansion up to degree and order 180, will be applied. Alternatively, the topography model GTM3a from H.-G. Wenzel, see [101], Wenzel 1998 and its given references, can be adopted. For the global potential field the GPM98b model from H.-G. Wenzel, cf. [101], [102] Wenzel 1998, will be used. The well known EGM96 model, [59] Lemoine 1998, can optionally replace the GPM98b model. In spite of the fact that the spherical harmonic models of H.-G. Wenzel have a maximum degree and order of 1800, the degree for representing the topography is usually set to 180. For the gravitational field of the Earth a representation up to degree 360 is generally sufficient for the intended purpose. At last, a brief summary concludes this chapter.

7-1 Numerical studies on the regular gravity space approach

This section aims at the numerical analysis of the regular gravity space approach introduced in Chapter 5. The planned modus operandi for the analysis has already been specified in Fig. 7.1. On the basis of the underlying diagram, it seems reasonable that the following investigations are elaborated in five separate paragraphs. As a start, a few introductory remarks are required, concerning the discretization of the data and the applied data grid for example. The next paragraph addresses the generation of the boundary surface. The third paragraph is devoted to establishing the corresponding boundary values and their analytical continuation. Thereafter, the BVP for the adjoint disturbing potential $\delta\psi$ is actually solved and a representation for $\delta\psi$ in terms of a spherical harmonics series expansion derived. Finally, after recomputing the surface of the Earth and/or the terrestrial gravitational potential, the numerical results for the closed-loop study are presented together with an assessment of the suitability of the investigated isotropic approach.
In the further course, the necessary work steps for the closed-loop study are explained individually in more detail within the subsequent paragraphs.

7-1.1 Preparatory considerations

According to the flowchart given in Fig. 7.1, the Earth's topography σ must first be established. Within the scope of an actual simulation study this must naturally be achieved point-wise. Practically, this discretization is associated with a certain data grid. Consequently, the horizontal positions (λ_i, ϕ_j) for the synthesis step are fixed and coincide with the known grid points of the specified grid.
For the considered purpose, the heights h_{ij}, which establish the Earth's surface σ, are computed for a $(i \cdot \Delta\lambda, \phi_j)$-grid. This grid is designed to meet the requirements of the subsequent Gaussian integration step. It is equally spaced along the parallels. Along the meridians, the spacing depends on the roots of the Legendre polynomial $P_{K+1}(\sin\phi)$. Consequently, it results that it exists symmetry with respect to the equator between the northern and the southern hemisphere and that the poles do not represent grid points themselves. The parameter K is related to the spectral resolution of the data grid. For example, if a simulated data set ought to be analyzed, which is based on a spherical harmonic series expansion up to degree and order $K = 767$, the minimum dimension of the data grid is predefined. More precisely, the locations of the 768 parallels must be determined as the roots of the Legendre polynomial $P_{768}(\sin\phi)$. Furthermore, 1536 evenly spaced meridians complement the grid. Under these conditions all spherical harmonic coefficients up to degree and order $K = 767$ can be uniquely determined. As a matter of fact, the specifications of this example have been chosen in agreement with the $(i \cdot \Delta\lambda, \phi_j)$-grid actually used within the conducted studies. See also Appendix C-2 for more details about the data grid and the corresponding integration step.
Hence, based on the known coefficients $\bar{c}_{kl}^{\mathrm{GTM}}$ of the selected GTM, the topographical heights h_{ij} are synthesized by evaluating the following spherical harmonic series expansion, see also (2.56) and (2.57),

$$h_{ij} = h(i \cdot \Delta\lambda, \phi_j) = \sum_{k=0}^{K} \sum_{l=-k}^{k} \bar{c}_{kl}^{\mathrm{GTM}} \bar{Y}_{kl}(i \cdot \Delta\lambda, \phi_j) \,. \tag{7.1}$$

Remark 33 With regard to (7.1), some thoughts have to be given concerning the following practical aspects. The direct use of the TUG87 coefficients set of M. Wieser for the height information synthesis according to (7.1) leads, in the first instance, to orthometric heights, which in the present studies will be used as heights, e.g., above the GRS80 ellipsoid. However, in view of (7.2), heights above a sphere are required. For this reason, a conversion of the original TUG87 coefficients by means of a spherical harmonic re-analysis of real heights above the sphere must be conducted in order to obtain an appropriate set of coefficients. This intermediate step is mandatory to directly obtain heights above the sphere from evaluating (7.1). In the further course of the research, proper consideration of this peculiarity is implicitly assumed.
Moreover, in the case of the original GTM3a model of H.-G. Wenzel, the heights computed for the oceanic areas refer to the sea floor topography. In a first approximation, by simply setting all negative height values to zero and by re-analyzing the revised set of heights, a new set of model coefficients can be obtained, which provides zero-heights for the oceanic areas. As before, it is assumed that this ambiguity in topography models is properly accounted for.

Next, based on (7.1) and the known geopotential coefficients $\bar{c}_{kl}^{\mathrm{GPM}}$ of the selected GPM, surface potential values v_{ij} are generated by evaluating the familiar spherical harmonic series expansion, cf. (2.55)

$$v_{ij} = V(i \cdot \Delta\lambda, \phi_j, h_{ij}) = \frac{GM}{(R + h_{ij})} \left(1 + \sum_{k=2}^{K} \sum_{l=-k}^{k} \bar{c}_{kl}^{\mathrm{GPM}} \left(\frac{R}{R + h_{ij}} \right)^k \bar{Y}_{kl}(i \cdot \Delta\lambda, \phi_j) \right) . \quad (7.2)$$

At last, taking (7.2) into account, ground gravitational acceleration data $\tilde{\mathbf{g}}_{ij}$ is derived according to

$$\tilde{\mathbf{g}}_{ij} = \tilde{\mathbf{g}}(i \cdot \Delta\lambda, \phi_j) = \nabla V(i \cdot \Delta\lambda, \phi_j, h_{ij}) . \quad (7.3)$$

This provides the input data in a gridded form, which is essentially required for another two synthesis steps coming next.

7-1.2 Synthesis of the boundary surface

It is now possible, based on the data acquired from (7.3) and by utilizing the transformation relationship given in Definition 20 in Section 5-1, to determine the boundary surface Σ in a point-wise manner

$$\boldsymbol{\xi}_{ij}\big|_{\Sigma} = -\sqrt{GM} \frac{\tilde{\mathbf{g}}_{ij}}{\|\tilde{\mathbf{g}}_{ij}\|^{3/2}} . \quad (7.4)$$

Alternatively, especially suitable for checking purposes, the boundary surface Σ can be deduced point-wise, as discussed in Lemma 17 and Definition 21 in Section 5-1.2, by solving the equations

$$\nabla V_0(\boldsymbol{\xi}_{ij}\big|_{\Sigma}) = \tilde{\mathbf{g}}_{ij} \quad (7.5)$$

with Newton's iteration method, see Appendix C-1, for $\boldsymbol{\xi}_{ij}\big|_{\Sigma}$. The above condition that the gradient of the gravitational potential V at the physical surface of the Earth, i.e. $\tilde{\mathbf{g}}$, must balance the gradient of the normal potential ∇V_0 at the boundary surface Σ, is already familiar from Definition 18 in Section 4-5. As a result, the boundary surface Σ, despite being defined in gravity space, can also be denoted by *gravimetric telluroid* as has been discussed before in Section 5-1.2.

Moreover, from (7.5) an essential property of the gravimetric telluroid Σ is easy to comprehend. As is known, the isotropic normal potential V_0 represents only a first approximation of the actual gravitational potential V. That is, $V_0 \approx V$ with an relative accuracy of 10^{-3} applies. Therefore it must be expected that the gravimetric telluroid Σ deviates from the Earth's surface σ by the same order of magnitude. As a result, significant separations between the gravimetric telluroid Σ and the topography σ of up to 20 km are imaginable. This is confirmed by Fig. 7.2, where the resulting differences of the surfaces Σ and σ are displayed.

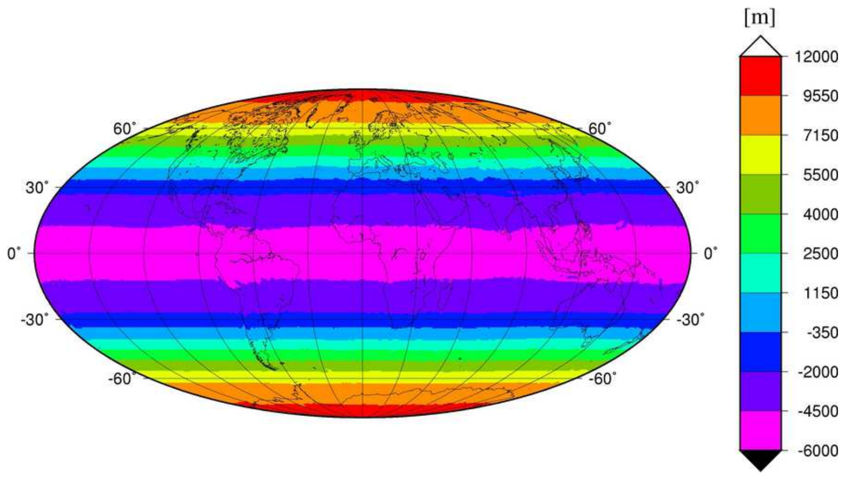


Figure 7.2: Separation between the gravimetric telluroid and the Earth's topography.

For the equatorial to the lower mid-latitude regions, the plotted values are negative. This means that the gravimetric telluroid runs below the Earth's surface. Consequently, towards the poles the gravimetric telluroid is obviously located above the Earth's surface. Due to the fact that Fig. 7.2 exhibits a uniform gradation from the equator towards the poles, the following can be presumed. As expected, since the magnitude of the ellipsoidal effect is much larger than the magnitude of the topographical effect, the gravimetric telluroid follows only along general lines the shape of the Earth's surface and no topographical features are identifiable at all. In other words, the application of a solely isotropic reference potential, thus disregarding the flattening of the Earth's gravity field, leads to the observed dominating elliptical trend. Interestingly, the oblateness of the gravimetric telluroid is smaller than for the Earth's surface. Hence, the actual shape and the detailed structure of the boundary surface Σ can only be revealed by plotting the deviations of the gravimetric telluroid with respect to a best-fitting ellipsoid of revolution E_Σ, as has been done in Fig. 7.3. The corresponding parameters of E_Σ, i.e. semi-major axis a_Σ and flattening f_Σ, can be determined, for example, by least-squares methods from the following system of equations

$$r_{\Sigma_{ij}} + v_{ij} = a_\Sigma(1 - f_\Sigma \sin^2 \phi_i)\,, \tag{7.6}$$

see [67] Moritz 1990, where $r_{\Sigma_{ij}}$ is known from

$$r_{\Sigma_{ij}} = \|\boldsymbol{\xi}_{ij}\| \quad , \quad \boldsymbol{\xi}_{ij} \in \Sigma\,. \tag{7.7}$$

Typically, the following parameters are estimated for E_Σ

$$\begin{aligned} a_\Sigma &= 6372993\,\text{m} \\ b_\Sigma &= 6368897\,\text{m} \\ f_\Sigma &= 0.000800 \\ e^2_\Sigma &= 0.001599\,. \end{aligned} \tag{7.8}$$

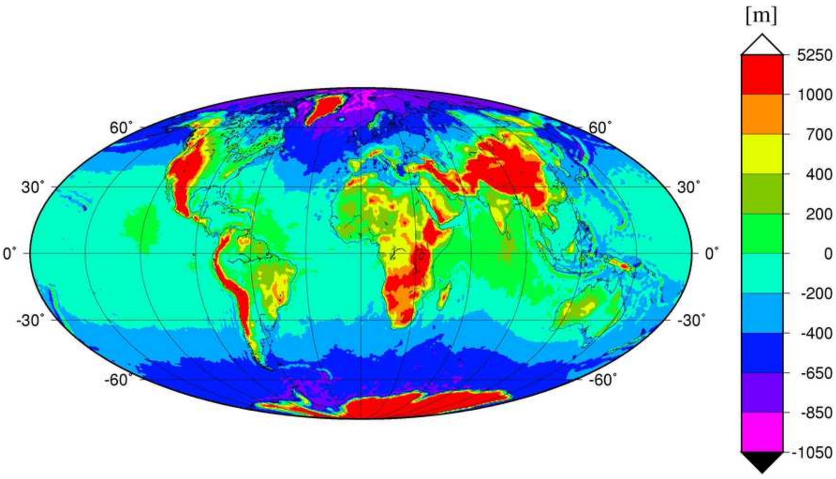


Figure 7.3: The gravimetric telluroid wrt. a mean ellipsoid.

In Fig. 7.3 the differences between Σ and E_Σ vary from -1 km to 5 km, which reflects both the spatial variation of the potential V and the variation of the topographic heights h, though the prevailing influence on Σ is clearly the topography. This supports the above reasoning that the form of the gravimetric telluroid primarily reproduces the form of the Earth's surface. Moreover, the estimates (7.8) for the mean ellipsoid E_Σ indicate that the boundary surface Σ in gravity space indeed features a lower flattening than a globally best-fitting Earth ellipsoid and, consequently, than the Earth's surface σ itself. E.g., within the framework of the Geodetic Reference System 1980 (GRS80), the basic parameters are specified as follows

$$\begin{aligned} a &= 6378137\,\text{m} \\ b &= 6356752\,\text{m} \\ f &= 0.003352 \\ e^2 &= 0.006694\,. \end{aligned} \tag{7.9}$$

In summary it can be said that, on the one hand, using an isotropic normal potential as the linearization point for the true potential leads to significant separations of the gravimetric telluroid and the physical Earth's surface of up to 12 km. Yet on the other hand, the reduced flattening of Σ can be of importance for the subsequent continuation process of the boundary data. As will be explained in more detail in the next section, the boundary values must be transfered from the gravimetric telluroid Σ to the Brillouin sphere S. For that purpose, the distance between Σ and S is outlined in the next figure.

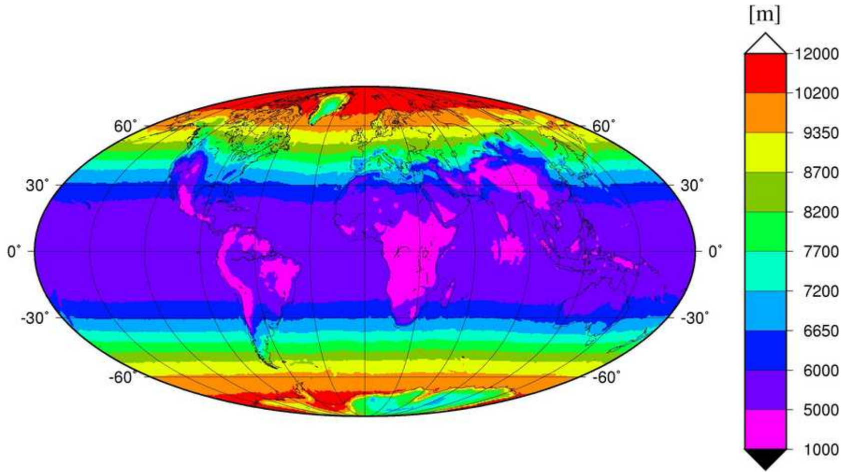


Figure 7.4: Separation of the gravimetric telluroid and the Brillouin sphere.

In contrast to the Earth's surface, which deviates from the Brillouin sphere by about 22 km at the poles, see e.g. [88] Seitz 1997, the maximum deviation of the gravimetric telluroid from the same sphere is in the order of only 12 km. The resulting consequences will be commented on later.

In a next step, a particular characteristic of the gravimetric telluroid is to be elaborated on. As described earlier in Section 7-1.1, the input data, i.e. h_{ij}, v_{ij} and $\tilde{\mathbf{g}}_{ij}$ are generated on the specific $(i \cdot \Delta\lambda, \phi_j)$-grid required for the subsequent spherical harmonic analysis step. Unfortunately, the determination of the boundary surface Σ according to (7.4) or (7.5) does not necessarily maintain the distribution of the telluroid points according to the lattice formation. This is documented by the next two figures given below, where the horizontal displacements of the telluroid points with respect to the nodal points of the data grid are illustrated.

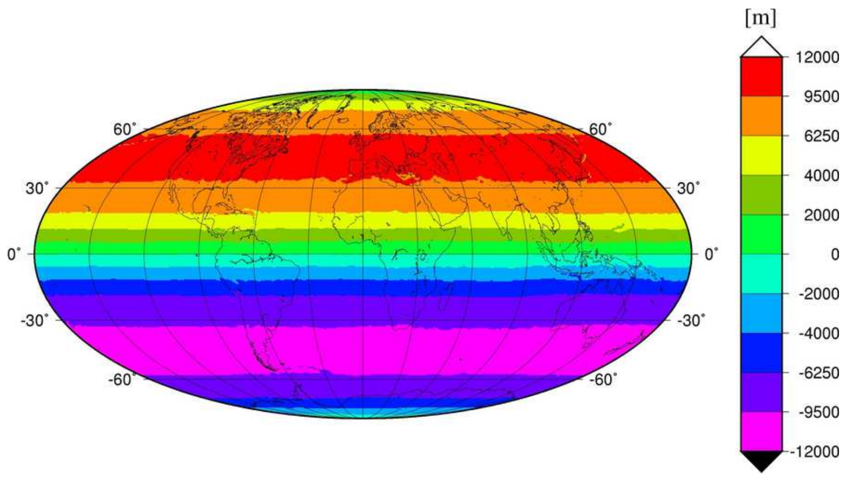


Figure 7.5: North-South displacements of the gravimetric telluroid points wrt. the grid points.

In Fig. 7.5, positive values reflect a shift of the actual telluroid points compared to the nodal points of the data grid into northern direction. Accordingly, negative values correspond to a displacement in southern direction. Consequently, on the northern as well as on the southern hemisphere, a displacement of the telluroid points towards the poles can be identified with the largest dislocations in the mid-latitude areas. This characteristic directly opposes the situation portrayed in Fig. 7.2. There, the telluroid has been observed to approach the Earth's surface closest in the mid-latitude regions. This means that in the mid-latitude bands, where the surfaces Σ and σ roughly coincide, the difference between the true potential V and the reference potential V_0 is absorbed by a horizontal, mainly latitudinal, telluroid displacement.

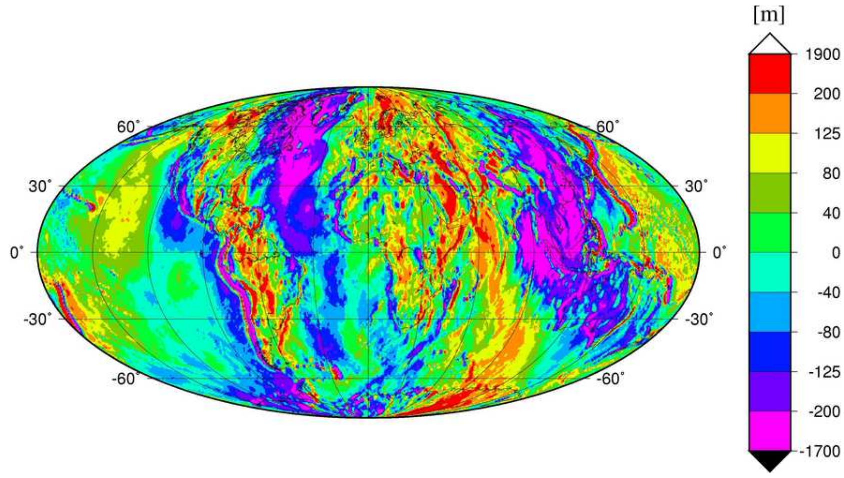


Figure 7.6: East-West displacements of the gravimetric telluroid points wrt. the grid points.

As a matter of fact, see Fig. 7.6, it can be stated that the longitudinal telluroid shift is of considerably smaller magnitude. In Fig. 7.6, the eastern displacement of telluroid points is indicated by positive values, whereas negative numbers mark a western shift. In contrast to the previous result concerning the latitudinal variations of actual and target locations, a correlation of the longitudinal variations and the topography σ can be pointed out.

The recovered telluroid offset between actual and target grid locations at the order of magnitude of several kilometers is relevant for the subsequent GSHA step. Fortunately, the simulation study environment provides the possibility to approximately synthesize the gravimetric telluroid Σ in conformity with the desired grid points and to assess the influence of the discussed horizontal displacement.

The following lemma addresses the issue of determining $\boldsymbol{\xi}_{ij}\big|_\Sigma = r_{\xi_{ij}} \left[\begin{array}{ccc} \cos\lambda_i \cos\phi_j & \sin\lambda_i\cos\phi_j & \sin\phi_j) \end{array} \right]^\top$, cf. (5.108), in accordance with the $(i\cdot\Delta\lambda, \phi_j)$-grid introduced in the last section:

Lemma 43 *The boundary surface Σ, realized point-wise in the nodal points $(i\cdot\Delta\lambda, \phi_j)$ of the data grid, results from*

$$r_{\xi_{ij}} = r - \frac{R}{2}\sum_{k=2}^{K}\sum_{l=-k}^{k}(k+1)\bar{c}_{kl}\left(\frac{R}{r}\right)^{k-1}\bar{Y}_{kl}(\lambda_\xi,\phi_\xi) \tag{7.10}$$

subject to

$$r = (R + h_{ij}) \quad , \quad \lambda_\xi = (i\cdot\Delta\lambda) \quad , \quad \phi_\xi = \phi_j\,. \tag{7.11}$$

Proof. Starting from

$$\nabla V(\mathbf{x}) = -\frac{GM\boldsymbol{\xi}}{\|\boldsymbol{\xi}\|^3}\,, \tag{7.12}$$

which holds according to Lemma 17,

$$\nabla V(\mathbf{x}) \approx -\frac{GM\mathbf{x}}{\|\mathbf{x}\|^3} + [\nabla\nabla V_0(\mathbf{x})]\,(\boldsymbol{\xi}-\mathbf{x}) \tag{7.13}$$

is derived by expanding the right-hand side of (7.12) at $\mathbf{x}$ into a Taylor series. Then, re-arranging (7.13) yields

$$\begin{aligned}
\boldsymbol{\xi} &= \mathbf{x} + [\nabla\nabla V_0(\mathbf{x})]^{-1}\left(\nabla V(\mathbf{x}) + \frac{GM\mathbf{x}}{\|\mathbf{x}\|^3}\right)\\
&= \mathbf{x} - \frac{\|\mathbf{x}\|^3}{GM}\left[\mathbf{I} - 3\frac{\mathbf{x}\mathbf{x}^\top}{\|\mathbf{x}\|^2}\right]^{-1}\left(\nabla V(\mathbf{x}) + \frac{GM\mathbf{x}}{\|\mathbf{x}\|^3}\right)\\
&= \mathbf{x} - \frac{\|\mathbf{x}\|^3}{GM}\left[\mathbf{I} - \frac{3}{2}\frac{\mathbf{x}\mathbf{x}^\top}{\|\mathbf{x}\|^2}\right]\left(\nabla V(\mathbf{x}) + \frac{GM\mathbf{x}}{\|\mathbf{x}\|^3}\right)\\
&= \mathbf{x} - \frac{\|\mathbf{x}\|^3}{GM}\left(\nabla V(\mathbf{x}) + \frac{GM\mathbf{x}}{\|\mathbf{x}\|^3} - \frac{3}{2}\frac{\mathbf{x}\mathbf{x}^\top}{\|\mathbf{x}\|^2}\nabla V(\mathbf{x}) - \frac{3}{2}\frac{GM\mathbf{x}}{\|\mathbf{x}\|^3}\right)\,.
\end{aligned} \tag{7.14}$$

With regard to (7.14), summarizing the term in parentheses, adopting spherical coordinates and disregarding the partial derivatives V_λ, V_ϕ leads to

$$\begin{aligned} r_\xi &= r - \frac{r^3}{GM}\left(V_r(\lambda,\phi,r) - \frac{3}{2}V_r(\lambda,\phi,r) + \frac{1}{2}V_{0r}(\lambda,\phi,r)\right) \\ &= r + \frac{1}{2}\frac{r^3}{GM}\left(V_r(\lambda,\phi,r) - V_{0r}(\lambda,\phi,r)\right). \end{aligned} \tag{7.15}$$

At last, insertion of (2.66) in (7.15) gives

$$r_\xi = r - \frac{R}{2}\sum_{k=2}^{K}\sum_{l=-k}^{k}(k+1)\bar{c}_{kl}\left(\frac{R}{r}\right)^{k-1}\bar{Y}_{kl}(\lambda,\phi),$$

and in particular for the designated grid

$$r_{\xi_{ij}} = (R+h_{ij}) - \frac{R}{2}\sum_{k=2}^{K}\sum_{l=-k}^{n}(k+1)\bar{c}_{kl}\left(\frac{R}{R+h_{ij}}\right)^{k-1}\bar{Y}_{kl}(i\cdot\Delta\lambda,\phi_j). \qquad \diamondsuit$$

In the following, the gravimetric telluroid determined according to the rigorous definition, (7.4), can be compared point-wise to the gridded telluroid realization established on the basis of Lemma 43. In foresight of the boundary data generation given in the next section this comparison is best accomplished by plotting the differences in radial distance of the corresponding telluroid points. The reason for observing only the radial variations is justified in consideration of (5.99) and Theorem 7. That is, in order to compute the reference or normal boundary values $v_0 = V_0\big|_\Sigma$ only the magnitude $\|\boldsymbol{\xi}\|$ is required. Accordingly, in Fig. 7.7 the differences $\|\boldsymbol{\xi}_{ij}\big|_\Sigma\| - r_{\xi_{ij}}$, cf. (7.4) and (7.10), are illustrated.

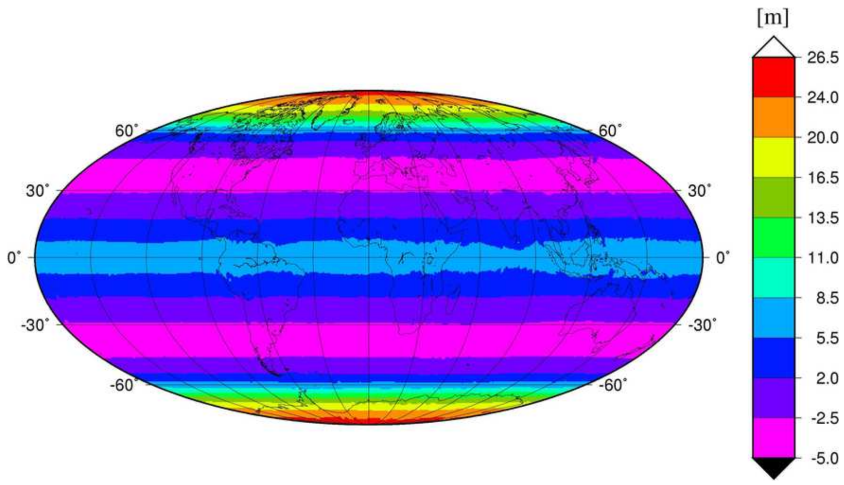


Figure 7.7: Differences in radial direction between the rigorous and the gridded telluroid realization.

As can be seen in Fig. 7.7, the telluroid differences shown above exhibit magnitudes of up to roughly 25 m. These maxima are reached at the poles. For the poles and the equatorial regions the true gravimetric telluroid lies above the telluroid that is realized in the grid locations. The best agreement can be observed for the mid-latitude zones. For these areas the true gravimetric telluroid also partly lies below the gridded telluroid realization. Finally, even though it ought to be stressed once more that the gridded telluroid realization according to the previous lemma represents an approximation as well, the following can be claimed. Due to the distortion between the default data grid in ordinary space and the resulting data grid in gravity space as demonstrated in Figs. 7.5, 7.6 and 7.7, a significant degrading effect on the closed-loop results must be expected. In other words, the negligence of the lateral influence results in a considerable radial error.

7-1.3 Synthesis of the boundary data

After having obtained a point-wise realization of the boundary surface Σ in the previous section, the point-wise synthesis of the corresponding boundary values is addressed next. In fact, based on (5.99), thereby taking (7.2) and (7.4) into account, it holds

$$\Delta v_{ij} = v_{ij} - v_{0ij} = v_{ij} - \left.\frac{GM}{\|\boldsymbol{\xi}_{ij}\|}\right|_{\Sigma} . \tag{7.16}$$

Under the mapping of (5.1), the point-wise potential anomaly data Δv_{ij} represent the boundary values in gravity space on Σ, or rather, in the individual telluroid points $\boldsymbol{\xi}_{ij}\big|_{\Sigma}$, see also Definition 23. These potential anomalies are shown in Fig. 7.8.

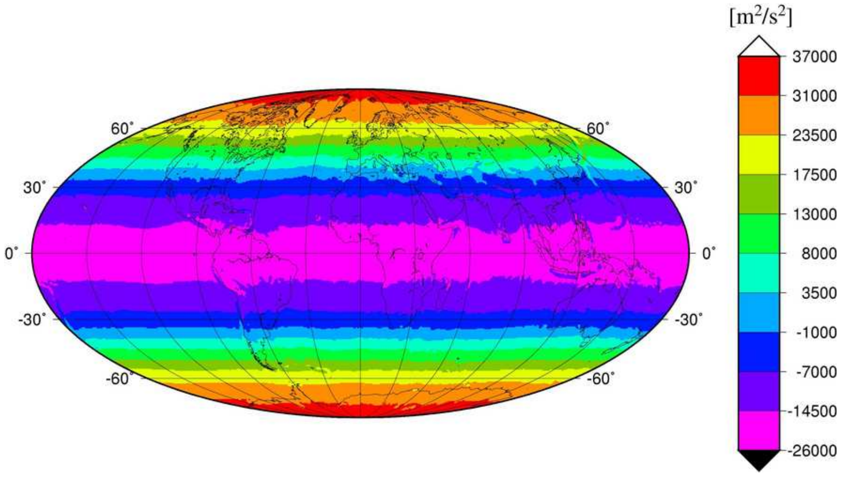
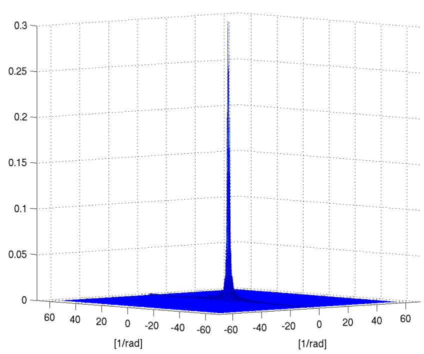


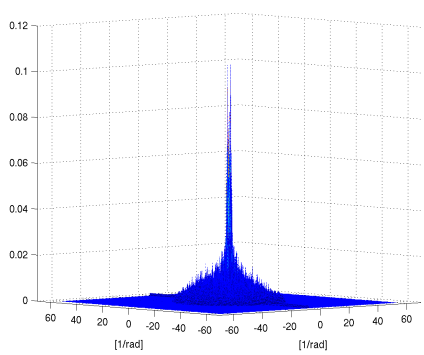
Figure 7.8: Boundary values.

In the first instance, the similarity between Fig. 7.2 and Fig. 7.8 stands out. As expected, sign and magnitude of the boundary values Δv_{ij} are correlated with the distance between the gravimetric telluroid Σ and the Earth's topography σ. Furthermore, it should be noted that due to the large separation of Σ and σ the topographical structure is hardly reflected in the boundary values.

At this point a short side note on the spectral behavior of the boundary surface and the associated boundary values is useful. As elaborated, e.g., in [18] Heck 1986 the gravimetric telluroid represents in general a rougher surface than the Marussi telluroid. The reason for that evolves from a comparison between Definition 6 and Definition 21. Evidently, potential gradient information is needed for the establishment of the gravimetric telluroid, which is naturally rougher than the potential information itself as required for the setup of the Marussi telluroid. On the other hand, potential anomalies are associated with the gravimetric telluroid, whereas according to Theorem 2 gravity anomalies are assigned to the Marussi telluroid. The assumption of smoother potential and rougher gravity data is supported by the next two diagrams, where the Fourier spectra of potential and gravity anomalies are compared to each other.



(a) potential anomalies



(b) gravity anomalies

Figure 7.9: Comparison of the 2d Fourier spectra of potential anomalies and gravity anomalies.

The Fourier spectra of both data types displayed in Fig. 7.9 confirm that the potential anomalies are indeed smoother than the gravity anomalies. As a matter of fact, this finding can also be adopted for the above mentioned telluroid definitions. Overall it might be concluded that the spectral influence on the solution of the GBVP is expected to be the same, when instead of the classical combination of Marussi telluroid and gravity anomalies, the combination gravimetric telluroid and potential anomalies is applied. Beyond it, for the actual continuation process of the boundary values, an increased smoothness in the boundary data should be beneficial. Hence, the possible use of potential anomalies might even be superior to the use of gravity anomalies.

Since just mentioned above, the aspect of data continuation will be addressed next. According to Definition 25 stated in Section 5-5, the boundary data Δv_{ij} is required for analysis reasons on a mathematically simple surface such as the sphere. The most straightforward method is to simply assume the values Δv_{ij} to be given on a sphere. That is,

$$\Delta v_{ij}\big|_S = \Delta v_{ij}\big|_\Sigma \tag{7.17}$$

can be regarded as a one-to-one projection of the potential anomalies Δv_{ij} from the gravimetric telluroid Σ onto a sphere S, e.g. onto the enclosing Brillouin sphere specified by $R = a_{EGM} = 6378136.3\,\text{m}$. Of course it should be kept in mind that in the case under consideration the separation between the gravimetric telluroid and the Brillouin sphere is in the order of several kilometers, cf. Figure 7.4. Hence, the question whether the idea of only projecting the data from the telluroid to the sphere is feasible or whether the projection error cannot be disregarded, must be carefully investigated. In the latter case, the use of an explicit data continuation procedure ought to be taken into account.

Remark 34 Another important issue related to the boundary value projection in terms of (7.17) should not be left unattended. As elaborated in the last section, the real telluroid positions differ significantly from the predestined nodal points, cf. Figs. 7.5 and 7.6. By definition, the boundary data established point-wise according to (7.16) refer to the actual telluroid points. However, within the scope of the intended harmonic analysis on the sphere the boundary information is required with respect to the special $(i \cdot \Delta\lambda, \phi_j)$-grid. Fortunately, the relationship (7.17) offers the opportunity to map the boundary values not only vertically onto the sphere but also horizontally from the actual telluroid points to the designated grid locations. Henceforth, the further proceeding is based on the assumption that the boundary values are given in accordance with the computational grid. It should be pointed out that this assumption is erroneous.

Similarly to the considerations conducted in conjunction with Lemma 43, the inaccuracy in the boundary data resulting from the reasoning outlined in the above remark can be roughly estimated, at least, within the framework of the underlying simulation environment. For this purpose, instead of the rigorous telluroid definition according to (7.4), the telluroid as defined by (7.10), i.e. in its realization directly for the nodal points, is used in (7.16). As a result, Fig. 7.10 illustrates the differences between the two boundary data variants.

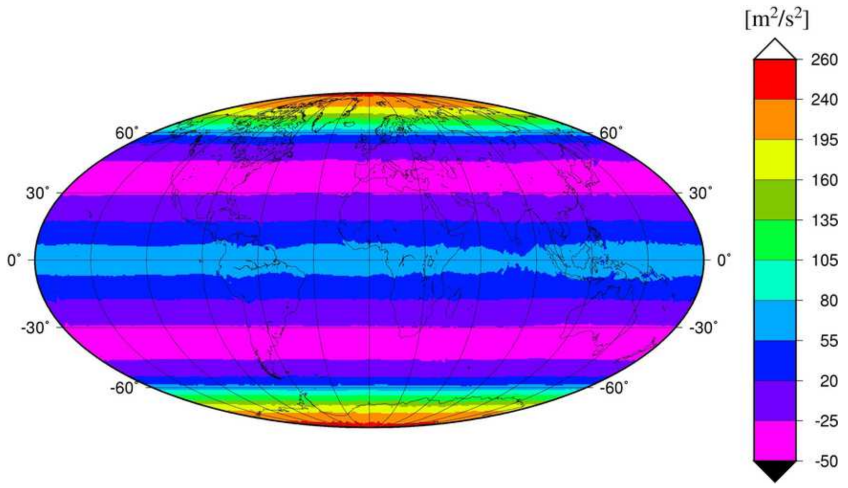


Figure 7.10: Differences in the boundary values in the actual telluroid points and in the designated grid points.

First of all, with regard to the boundary data variations presented in Fig. 7.10, it turns out that the overall behavior is of course strongly correlated to the telluroid differences documented before in Fig. 7.7. Again, the deviance is largest at the poles and smallest for the mid-latitude regions. In addition, the fact that in both cases the particular differences are not equally distributed in respect of positive and negative incidences should be recorded. If anything, a trend to positive differences can be claimed, which implies an overestimation of the actual boundary data with respect to target values in the grid points.

At last, the basic input data, synthesized along a test meridian in the Himalaya region, are compiled in figure given below to facilitate inspection.

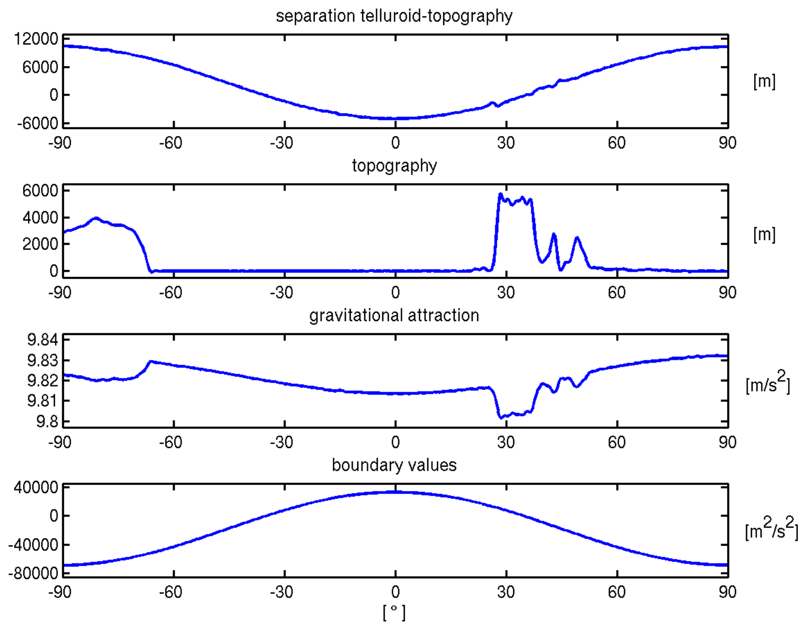


Figure 7.11: Profiles of the relevant quantities in the Himalayas.

As argued before, it can be concluded that within the presented isotropic approach neither the gravimetric telluroid nor the boundary values themselves are especially sensitive to the topographical signal. Furthermore, attention is called to the dimension of the separation between telluroid and topography and to the magnitude of the boundary data.

7-1.4 Numerical BVP solution

After having provided in the context of the previous two sections the synthetic input values in a point-wise form for a specific data grid on the sphere, the corresponding BVP in constant radius approximation will finally be solved numerically in the present section. More precisely, the adjoint disturbing potential is determined from the BVP introduced in Definition 25. For this purpose, the problem is considered in the spectral domain and, consequently, solved by means of a spherical harmonic analysis procedure, which is described in detail in Appendix C-2.

As a start, the underlying BVP is recapitulated for the sake of convenience as follows

$$\Delta\delta\psi(\boldsymbol{\xi}) = 0\,, \quad \|\boldsymbol{\xi}\| > R \tag{7.18}$$

$$-\left(\frac{1}{2}r_\xi\frac{\partial\delta\psi}{\partial r_\xi} + \delta\psi\right)\Bigg|_S = \Delta v\,. \tag{7.19}$$

As is generally known, the following spherical harmonic series expansion

$$\delta\psi(\lambda_\xi,\phi_\xi,r_\xi) = \frac{1}{R}\sum_{n=0}^{\infty}\sum_{m=-n}^{n}\bar{\tau}_{nm}\left(\frac{R}{r_\xi}\right)^{n+1}\bar{Y}_{nm}(\lambda_\xi,\phi_\xi)\,, \tag{7.20}$$

cf. Section 2-2.4, already constitutes a solution of Laplace's partial differential equation (7.18), which is subject to

$$r_\xi = R \quad , \quad \lambda_\xi = (i\cdot\Delta\lambda) \quad , \quad \phi_\xi = \phi_j \tag{7.21}$$

according to the boundary condition (7.19).

Moreover, by means of a global spherical harmonic analysis procedure, see Appendix C-2, the coefficients of a spherical harmonics series expansion for the potential anomalies that are given on the sphere can be determined. Hence, in the spectral domain the representation of the potential anomalies on the sphere reads as follows

$$\Delta v(\lambda_\xi, \phi_\xi) = \sum_{k=0}^{\infty} \sum_{l=-k}^{k} \bar{c}_{kl} \bar{Y}_{kl}(\lambda_\xi, \phi_\xi)\,. \tag{7.22}$$

Now, the unknown coefficients $\bar{\tau}_{nm}$ can be deduced by insertion of (7.20) and (7.22) into the boundary condition (7.19). This is accomplished within the scope of the following lemma:

Lemma 44 *The adjoint disturbing potential, when expressed in terms of the following series expansion*

$$\delta\psi(\lambda_\xi, \phi_\xi, r_\xi) = \sum_{k=0}^{K} \sum_{l=-k}^{k} \frac{2\bar{c}_{kl}}{k-1} \left(\frac{R}{r_\xi}\right)^{k+1} \bar{Y}_{kl}(\lambda_\xi, \phi_\xi) \quad ; \quad k \neq 1\,, \tag{7.23}$$

constitutes a unique solution for the boundary value problem specified by (7.18) and (7.19), if the coefficients $\bar{c}_{kl}$ belong to the spherical harmonics series expansion (7.22) of the potential anomalies $\Delta v(\lambda_\xi, \phi_\xi)$ given on S.

Proof. Prior to the application of (7.20) and (7.22) to the boundary condition (7.19), a further preliminary consideration is useful. In view of (7.19),

$$\begin{aligned} \frac{\partial \delta\psi}{\partial r_\xi} &= \frac{\partial}{\partial r_\xi} \left(\frac{1}{R} \sum_{n=0}^{\infty} \sum_{m=-n}^{n} \bar{\tau}_{nm} \left(\frac{R}{r_\xi}\right)^{n+1} \bar{Y}_{nm}(\lambda_\xi, \phi_\xi) \right) \\ &= -\frac{1}{R}\frac{1}{r_\xi} \sum_{n=0}^{\infty} \sum_{m=-n}^{n} \bar{\tau}_{nm}(n+1) \left(\frac{R}{r_\xi}\right)^{n+1} \bar{Y}_{nm}(\lambda_\xi, \phi_\xi) \end{aligned} \tag{7.24}$$

is derived by taking (7.20) into account. Next, using (7.24) together with (7.20) and (7.22) in (7.19) yields

$$\begin{aligned} -\left(\frac{1}{2} r_\xi \frac{\partial \delta\psi}{\partial r_\xi} + \delta\psi\right)\Bigg|_S &= \Delta v \\ -\left(-\frac{1}{2R} \sum_{n=0}^{\infty} \sum_{m=-n}^{n} \bar{\tau}_{nm}(n+1) \left(\frac{R}{r_\xi}\right)^{n+1} \bar{Y}_{nm} + \frac{1}{R} \sum_{n=0}^{\infty} \sum_{m=-n}^{n} \bar{\tau}_{nm} \left(\frac{R}{r_\xi}\right)^{n+1} \bar{Y}_{nm} \right)\Bigg|_S &= \sum_{k=0}^{\infty} \sum_{l=-k}^{k} \bar{c}_{kl} \bar{Y}_{kl}\,. \end{aligned} \tag{7.25}$$

Re-arranging the left hand side of (7.25) leads to

$$\left(\frac{1}{R} \sum_{n=0}^{\infty} \sum_{m=-n}^{n} \bar{\tau}_{nm} \left(\frac{n+1}{2} - 1\right) \left(\frac{R}{r_\xi}\right)^{n+1} \bar{Y}_{nm}(\lambda_\xi, \phi_\xi) \right)\Bigg|_S = \sum_{k=0}^{\infty} \sum_{l=-k}^{k} \bar{c}_{kl} \bar{Y}_{kl}(\lambda_\xi, \phi_\xi)\,. \tag{7.26}$$

In agreement with the underlying boundary condition,

$$\left(\frac{1}{R} \sum_{n=0}^{\infty} \sum_{m=-n}^{n} \bar{\tau}_{nm} \frac{n-1}{2} \bar{Y}_{nm}(\lambda_\xi, \phi_\xi) \right)\Bigg|_S = \sum_{k=0}^{\infty} \sum_{l=-k}^{k} \bar{c}_{kl} \bar{Y}_{kl}(\lambda_\xi, \phi_\xi) \tag{7.27}$$

is obtained from (7.26) by identifying $r_\xi = R$. Then, a comparison of the coefficients on both sides of (7.27) reveals

$$\bar{\tau}_{kl} = \frac{2R}{k-1} \bar{c}_{kl} \quad ; \quad k \neq 1\,. \tag{7.28}$$

Consequently, by taking advantage of (7.28) in (7.20), it finally results in terms of a limited series expansion

$$\delta\psi(\lambda_\xi, \phi_\xi, r_\xi) = \sum_{k=0}^{K} \sum_{l=-k}^{k} \frac{2\bar{c}_{kl}}{k-1} \left(\frac{R}{r_\xi}\right)^{k+1} \bar{Y}_{kl}(\lambda_\xi, \phi_\xi) \quad ; \quad k \neq 1\,. \qquad \diamond$$

Remark 35 The exclusion of $k = 1$ in (7.23) necessitates further clarification. From (7.25) and (7.27) directly results the following spherical harmonics series representation for the potential anomalies on the sphere

$$\Delta v\Big|_S = \frac{1}{R} \sum_{n=0}^{\infty} \sum_{m=-n}^{n} \bar{\tau}_{nm} \frac{n-1}{2} \bar{Y}_{nm}(\lambda_\xi, \phi_\xi)\,.$$

Note, that this relationship is to some extent similar to the series expansion of the gravity anomalies, which is given in [28] HEISKANEN&MORITZ 1967 within the scope of the classical approach. In this context it is worth mentioning that the gravity anomalies were found to be free of degree $k = 1$ terms. This is considered to hold true in the present case as well. In fact, within the conducted spherical harmonic analysis, where the full spectrum of coefficients starting from degree $k = 0$ is determined, the degree $k = 1$ coefficients were found to be negligible for the overall result. More precisely, they turned out to be two orders of magnitude smaller than the other coefficients and, as far as the numerical realization is concerned, the omission due to the requirement $k \neq 1$ induces differences at the decimeter level. Compared to a total closed-loop error, which is in the range of several meters as will be calculated in the next section, this effect is tolerable. Accordingly, within the achievable numerical accuracy the consistency condition $\bar{c}_{1l} = 0$ is considered to be satisfied.

As a result, the adjoint potential potential is obtained in view of (5.82) as the sum of the adjoint normal potential and the adjoint disturbing potential

$$\begin{aligned} \psi(\lambda_\xi, \phi_\xi, r_\xi) &= \psi_0(r_\xi) + \delta\psi(\lambda_\xi, \phi_\xi, r_\xi) \\ &= -2\frac{GM}{r_\xi} + 2\sum_{k=0}^{K}\sum_{l=-k}^{k} \frac{\bar{c}_{kl}}{k-1}\left(\frac{R}{r_\xi}\right)^{k+1} \bar{Y}_{kl}(\lambda_\xi, \phi_\xi)\,. \end{aligned} \tag{7.29}$$

Once again, it must be pointed out that the above utilized series expansion coefficients $\bar{c}_{kl}$ are related to the harmonic analysis of the boundary data Δv, see also Lemma 44.

Additionally, according to (5.106), the height anomaly vector $\boldsymbol{\zeta}$ follows from (7.23). That is,

$$\boldsymbol{\zeta} = \gamma(\boldsymbol{\xi}|_\Sigma)\nabla_\xi \delta\psi(\boldsymbol{\xi}|_\Sigma) \tag{7.30}$$

holds true. See Section 2-2.4 for a similar example with regard to the evaluation of $\nabla_\xi \delta\psi(\boldsymbol{\xi})$. Consequently, by taking (5.105) into account, the surface of the Earth σ, constituted by the position vector

$$\mathbf{x}|_\sigma = \mathbf{x}|_\Sigma + \boldsymbol{\zeta}\,, \tag{7.31}$$

is obtained. The prior relationship finalizes the last step within the closed-loop study, cf. Fig 7.1. Hence, on the basis of the considerations given in the previous four sections, the associated numerical result will be given next.

7-1.5 Result of the closed-loop study

This section yields the numerical result of the closed-loop study for the approach given in Chapter 5. The setup of the study has been explicated in detail in the last sections. The corresponding computations were conducted in compliance with the working plan illustrated by the flowchart given in Fig. 7.1. The final result in terms of a point-wise comparison of target and actual topography is shown in Fig. 7.12.

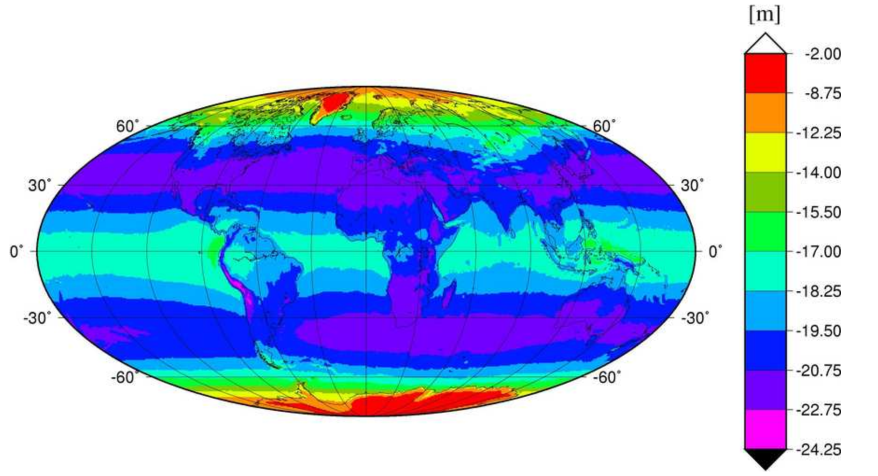


Figure 7.12: Closed-loop comparison of given and recovered topography.

First of all, it can be stated that the obtained result mostly complies with the expected result. That is, the overall consistency ranges only at the meter level. More precisely, as can be deduced from the table given below, which provides the statistical key figures for the closed-loop error, the mean radial variation between true and estimated topography points is about 17 m with peak errors of up to 25 m.

RMS [m]	mean [m]	std [m]	max [m]	min [m]
17.55	-16.93	4.62	-1.54	-24.23

Table 7.1: Statistics of the closed-loop comparison of given and recovered topography.

The considerable error size has several causes, which have already been identified in the previous sections. For one thing, large distortions between the actual data grid and the required $(i \cdot \Delta\lambda, \phi_j)$ analysis grid have been revealed in Section 7-1.2. As a consequence thereof, it has already been reckoned that the resulting influence on the boundary surface and on the boundary data induces a significant error, i.e. at the meter level, during the GSHA process. As a rule, errors due to this imperfection manifest themselves in the fact that the largest differences shown in Fig. 7.12 occur for the mid-latitude regions. For another thing, the above result indicates that an explicit data continuation is very likely indispensable. This can be understood against the background of (7.17), which specifies the mapping of the boundary data onto the computational sphere. As a matter of fact, a simple projection does not account for the actual behavior of the boundary values to diminish with increasing distance from the geocenter. Apparently, within the approach under investigation, the error when only projecting the data is too large. As a result, the average level of the projected boundary data is too high, which leads to the scenario in Fig. 7.12 that exhibits only negative values, i.e. that contains clearly an offset. The 2d Fourier analysis of the associated closed-loop topography differences presented above, which is provided in Fig. 7.21 together with the resulting closed-loop errors for the ellipsoidal approach, actually supports this finding by showing a distinct peak in the center of the spectral power diagram. This reasoning can be further encouraged by the following consideration. The pronounced topography in the areas of Greenland and Antarctica, as compared to the regions in the immediate vicinity, leads to a reduced separation between the topography and the bounding sphere and, relatively speaking, to a lower error level.

In conclusion, on the basis of the result obtained above, it can be claimed that in general the intended numerical proof of concept for the isotropic regular gravity space approach, Chapter 5, has been accomplished. However, it must be acknowledged that the achieved numerical accuracy is rather unsatisfactory. On this account, the numerical strategy to solve the underlying BVP must be improved. For example, instead of the projection method applied up to now the use of explicit data continuation in combination with a re-gridding procedure could be imagined. Additionally, an iterated solution process might be necessary. In any case, a substantial increase in complexity must be expected. The reason for this shortcoming is due to the much discussed fact that the observed difficulties are closely related to the inadequate choice of an isotropic reference potential within the linearization step. Consequently, one option to avoid cumbersome procedures such as explicit data continuation, re-gridding and iteration is to use the regular gravity space formulation introduced in Chapter 6. This will be confirmed in the following section, where the modified approach in gravity space based on a spheroidal linearization point is numerically investigated.

7-2 Numerical studies on the ellipsoidal regular gravity space approach

As already stated in the beginning, the second part of the this chapter refers to the numerical investigations accomplished on the basis of the modified regular approach presented in Chapter 6. In order to guarantee the comparability of both regular approaches, the setup of the upcoming closed-loop study remains practically the same and, consequently, complies with Fig. 7.1. For this reason, also the organization of the subsequent paragraphs is left unchanged. However, since most of the basic explanations and derivations supplied in the previous sections also hold true for the present context, the effort of repeating them once again can be avoided. Thus, if not stated otherwise, the further proceeding is on the assumption that the prior considerations are still valid.
Accordingly, instead of specifying the applied input models or discussing the aspect of data discretization in view of the specific analysis grid within the scope of the preparatory paragraph following next, a particular characteristic of the approach under investigation will be addressed. That is, the transformation into regular gravity space based

on a truncated series representation will be more closely investigated. Thereafter, synthesis and examination of the boundary surface and the boundary data are presented in the usual manner. As far as the numerical solution is concerned, the modus operandi outlined Section 7-1.4 remains virtually unaffected. The demonstration of the closed-loop error and a subsequent summary, comparing and assessing the achieved results of the examined regular approaches, conclude this chapter.

7-2.1 Preparatory considerations

As already announced, this paragraph aims to clarify the open issue about the accuracy that can be expected for the modified regular gravity space transformation according to Lemma 32 introduced in Section 6-1.1. Naturally, the transformation accuracy is a crucial factor to form an opinion on the overall quality level of the method. Furthermore, the intention is to contribute to a better understanding of a sui generis transformation, which is based on a series expansion representation. Thus, in view of Lemma 32 and by taking the specifications given in Section 7-1.1 into account, the following point-wise transformation relationship between ordinary and ellipsoidal regular gravity space can be recalled

$$\boldsymbol{\xi}_{ij}(\mathbf{p}_{ij}, J_2) = \boldsymbol{\xi}_0(\mathbf{p}_{ij}) + J_2\boldsymbol{\xi}_1(\mathbf{p}_{ij}) + J_2^2\boldsymbol{\xi}_2(\mathbf{p}_{ij}) + \mathrm{O}\big(J_2^3\big) \ , \tag{7.32}$$

which is subject to

$$\mathbf{p}_{ij} = \nabla V(i \cdot \Delta\lambda, \phi_j, r_{ij}) \, . \tag{7.33}$$

In order to satisfy the task of finding an accuracy estimate for the above transformation, the error when truncating the corresponding series representation, e.g. after the J_2^2-term, has to be determined. For that purpose the line of argument given in Section 6-1 to derive the above series representation has to be called to mind. In short, it has been the idea that the sought-after gravity space transformation must satisfy the identical mapping property, i.e. $\boldsymbol{\xi} = \mathbf{x}$, if the normal potential V_0^{ell}, see (6.7), is set equal to the actual potential V. Due to the fact that it was impossible to find a closed-form solution for the desired transformation, a representation in terms of the series expansion (7.32) had to be taken into account. Note that in the strict sense an infinite series is required to exactly satisfy the identity $\boldsymbol{\xi} = \mathbf{x}$. As a result of this, it is now possible in return to take advantage of the identical mapping property to estimate the truncation error of the above series representation. For that purpose, the application of the identical mapping condition for the case under investigation must be considered. More precisely, provided that

$$\tilde{\mathbf{p}}_{ij} = \nabla V_0^{ell}(i \cdot \Delta\lambda, \phi_j, r_{ij}) \tag{7.34}$$

holds true, the consistency of

$$\boldsymbol{\xi}_{ij}(\tilde{\mathbf{p}}_{ij}, J_2) \cong \mathbf{x}_{ij} \tag{7.35}$$

with

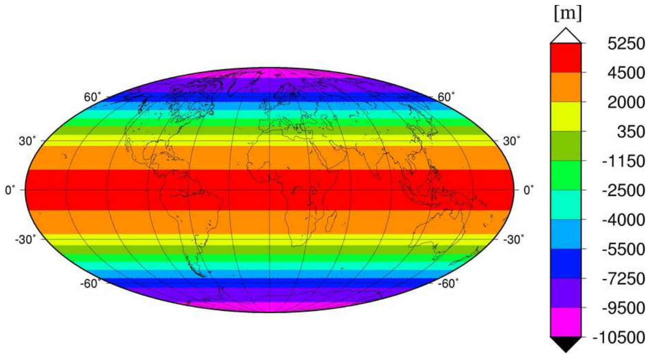
$$\mathbf{x}_{ij} = r_{ij} \left[\begin{array}{ccc} \cos(i \cdot \Delta\lambda)\cos\phi_j & \sin(i \cdot \Delta\lambda)\cos\phi_j & \sin\phi_j \end{array} \right]^\top \tag{7.36}$$

must be observed. It makes sense to accomplish this evaluation for surface points of the topography.

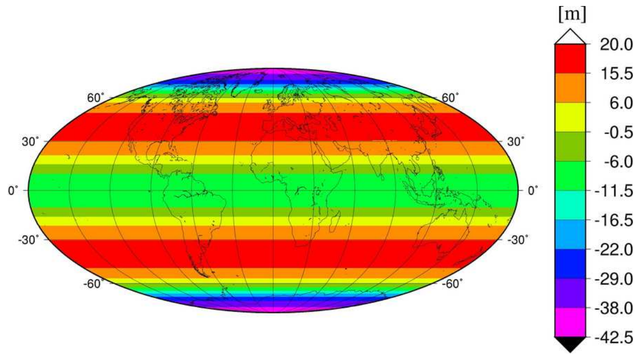
The corresponding result is illustrated by the figure given on the next page. The sequence of the three diagrams shown in Fig. 7.13 demonstrate the effect of truncating the series (7.32). In fact, the last plot refers thereby exactly to the second-order Taylor series approximation written out explicitly in (7.32). Consequently, the corresponding approximations of first and zero order are realized in the plots given in the middle and at the top. At the first glance, a tremendous improvement in consistency between input, i.e. $\mathbf{x}$, and output, i.e. $\boldsymbol{\xi}$, values can be constituted. To be more exact, whereas the error, cf. Fig. 7.13 (a), associated with the simplistic zero-order transformation

$$\boldsymbol{\xi}_{ij} = \boldsymbol{\xi}_0(\tilde{\mathbf{p}}_{ij}) \tag{7.37}$$

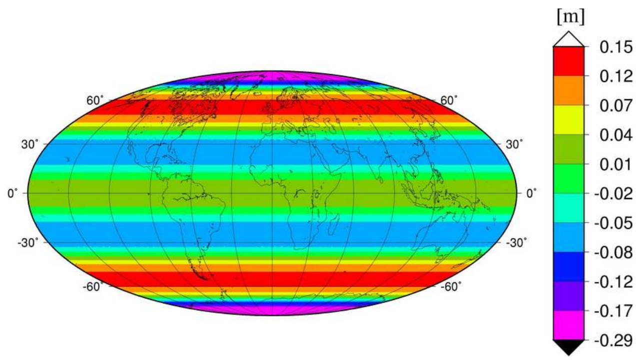
is in the order of magnitude of several kilometers, the inclusion of the first order term $J_2\boldsymbol{\xi}_1(\tilde{\mathbf{p}}_{ij})$ reduces the error down already to the multi-meter level, see Fig. 7.13 (b). Beyond it, Fig. 7.13 (a) referring to the above zero-order approximation is in conformity with Fig. 7.2, which is based on the afore investigated isotropic gravity space approach. This agreement is a confirmation of the current and the previous result. As a matter of fact, the zero-order approximation (7.37) and the transformation relationship (7.4) are comparable with each other. In both cases the flattening influence due to J_2 has been neglected. A subtle difference exists in such a way that, on the one hand, the effect of J_2 has been disregarded on the side of the normal potential and, on the other hand,



(a) zero order term only: J_2^0



(b) up to first order terms: J_2^1



(c) up to second order terms: J_2^2

Figure 7.13: Influence of series truncation on the separation between true and recovered topography.

on the side of the underlying transformation. Naturally, in view of (7.37) and (7.4), the consequences are the same. Furthermore, by accounting also for the influence of J_2^2, the truncation error drops in average below the decimeter as can be derived from Fig. 7.13 (c). This gives a valuable insight into the modified regular gravity space transformation (7.32) and provides the desired information about the behavior of the associated truncation error. To sum up, it can be claimed that the error of a few centimeters due to the truncation of the related series expansion starting from the terms of power J_2^3 can be tolerated, at least for the intended purpose of accomplishing a proof of concept for the considered approaches. Admittedly, the deduced truncation error level rests on the legitimate assumption that the error behavior remains the same if the normal potential V_0^{ell} instead of the true potential V is applied. However, it is of course always possible to increase the transformation accuracy by taking further higher order terms into account. For the sake of completeness it is worth mentioning that the considerations made in Section 7-1.1 with respect to the quantities h_{ij}, v_{ij}, $\tilde{\mathbf{g}}_{ij}$ continue to hold true.

7-2.2 Synthesis of the boundary surface

Similarly to Section 7-1.2, the formation of the boundary surface is the next task to address within the process of generating the simulation scenario. The synthesis of the boundary surface is conducted in view of Definition 28 given in Section 6-1.2. In addition, in taking the findings of previous section into account yields the relationship to establish the boundary surface in a point-wise manner

$$\boldsymbol{\xi}_{ij}\big|_{\Sigma} = \boldsymbol{\xi}_0(\tilde{\mathbf{p}}_{ij}) + J_2\boldsymbol{\xi}_1(\tilde{\mathbf{p}}_{ij}) + J_2^2\boldsymbol{\xi}_2(\tilde{\mathbf{p}}_{ij}) \tag{7.38}$$

with

$$\tilde{\mathbf{p}}_{ij} = \tilde{\mathbf{g}}_{ij} = \tilde{\mathbf{g}}(i\cdot\Delta\lambda, \phi_j) = \nabla V(i\cdot\Delta\lambda, \phi_j, h_{ij})\,, \tag{7.39}$$

which has analogy with (7.3).
The boundary surface Σ, which results according to the above specifications, is illustrated in Fig. 7.14 by plotting the differences of Σ with respect to the surface of the Earth σ.

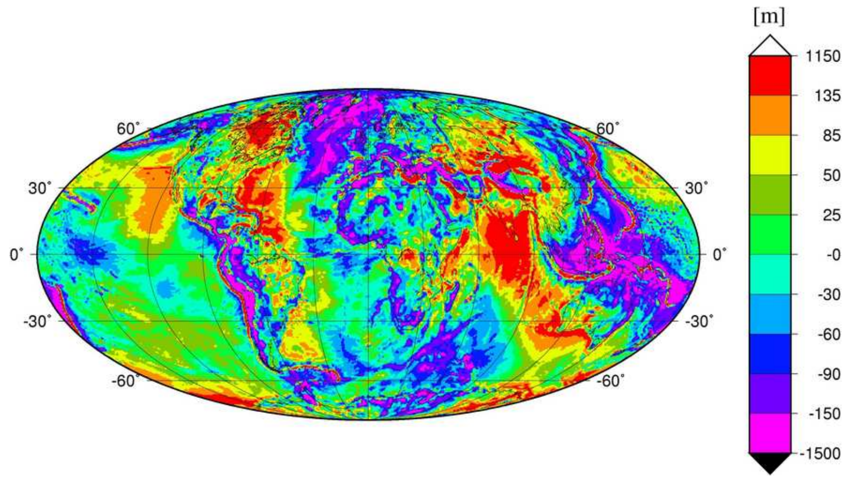


Figure 7.14: Separation between the boundary surface Σ and the Earth's topography.

As compared to Fig. 7.2, it can be clearly inferred from Fig. 7.14, that the surface Σ, when established on the basis of the ellipsoidal regular gravity space formulation, converges significantly to the topography σ. The surface Σ runs above the topography for positive numbers and below for negative ones. In fact, the mean separation between Σ and σ is about 78 m. This value is in the order of magnitude specified, e.g., in [18] Heck 1986 for the gravimetric telluroid under similar conditions. Hence, the completely new situation displayed in Fig. 7.14 complies with the main objective of the new regular approach. That is to say, the revised spheroidal linearization point leads indeed to the observed reduction of the position anomaly between the two surfaces Σ and σ. As a result, the formerly large distortions are expected, understandably enough, to decrease significantly due to the reduced spacing between Σ and σ. Indeed, this is confirmed by Figs. 7.15 and 7.16.

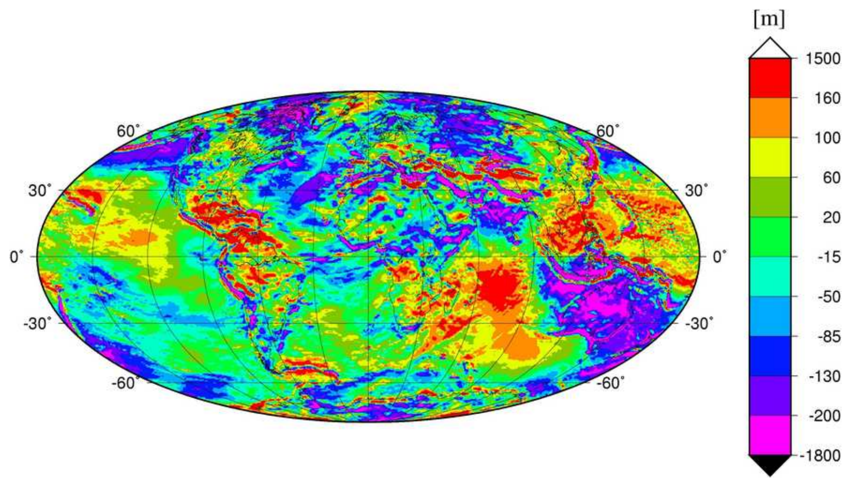


Figure 7.15: North-South displacements of the gravimetric telluroid points wrt. the grid points.

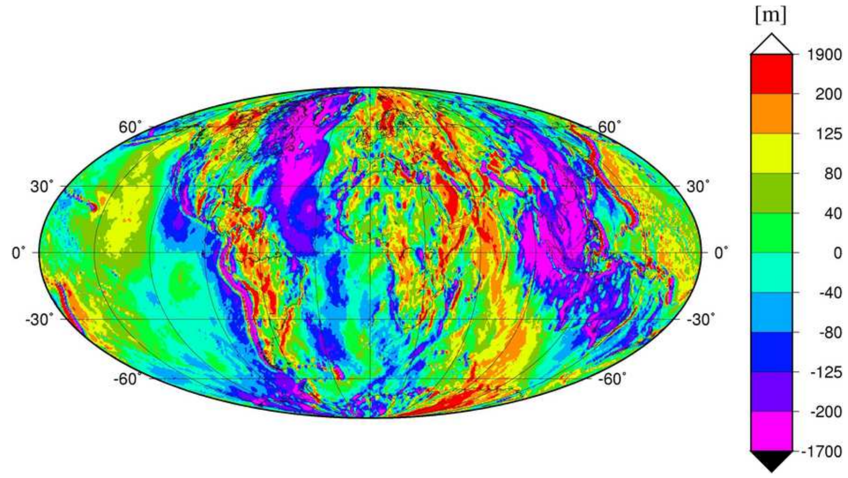


Figure 7.16: East-West displacements of the gravimetric telluroid points wrt. the grid points.

A comparison of Fig. 7.5 and Fig. 7.15 reveals a decline in the displayed distortions by about a factor of ten. More precisely, the latitudinal displacement field in the former case exhibits mean variations in the order of magnitude well above the kilometer level whereas in the latter case the differences come near the 100 m level. The longitudinal displacement fields, cf. Fig. 7.6 and Fig. 7.16, remain about the same, since neither the isotropic normal potential V_0 nor the modified reference potential V_0^{ell} depend on the longitude λ. Of course, it is assumed that this finding is beneficial for the later GSHA step.

Apart from that, another important parameter is the spacing between the boundary surface Σ and the Brillouin sphere S. As elaborated throughout the previous Section 7-1, the distance between Σ and S is crucial for the mapping of the boundary data from the actual boundary surface Σ to a better suited computational surface, i.e. to the Brillouin sphere S. Consequently, this quantity is portrayed in the figure given below.

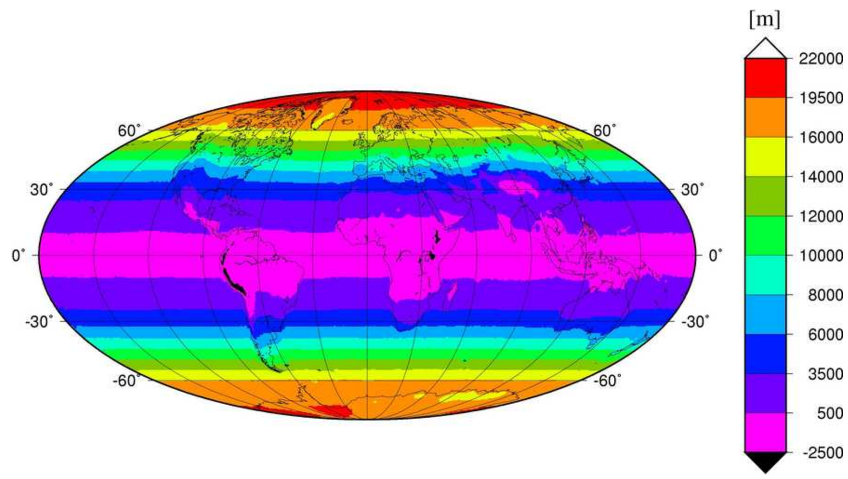


Figure 7.17: Separation of the boundary surface Σ and the Brillouin sphere S.

This result is not surprising since the surface Σ has been found to reproduce approximately the topographical surface σ. Hence, the situation shown in the above picture is owed to the fact that the Earth's surface σ itself deviates from the Brillouin sphere S in such a way. In fact, the discrepancy towards the poles of around 22 km is definitely associated with the flattening of Earth's body, see e.g. [88] Seitz 1997. However, in contrast to Fig. 7.4 it must be acknowledged that the separations between the boundary surface and the computational sphere observed above are about two times larger than before. Admittedly, this can be seen as a disadvantage of the modified approach as opposed to the prior inspected isotropic approach.

7-2.3 Synthesis of the boundary data

Comparable to Section 7-1.3, the prior generation of the underlying boundary surface is followed by the synthesis of the corresponding boundary values. These are acquired in terms of a point-wise representation that emanates from the right-hand side of (6.101) given in Section 6-3, i.e.

$$\Delta\bar{v}_{ij} = v_{ij} - \bar{v}_{0_{ij}} = v_{ij} - V_0^{ell}(i \cdot \Delta\lambda, \phi_j, \boldsymbol{\xi}_{ij})\Big|_{\Sigma} . \tag{7.40}$$

Regarding this expression, the surface potential values v_{ij} are still based on (7.2), whereas the reference potential values $\bar{v}_{0_{ij}}$ depend on the modified normal potential (6.8) and on the new boundary surface (7.38). The values $\Delta\bar{v}_{ij}$ resulting from (7.40) are visualized in Fig. 7.18.

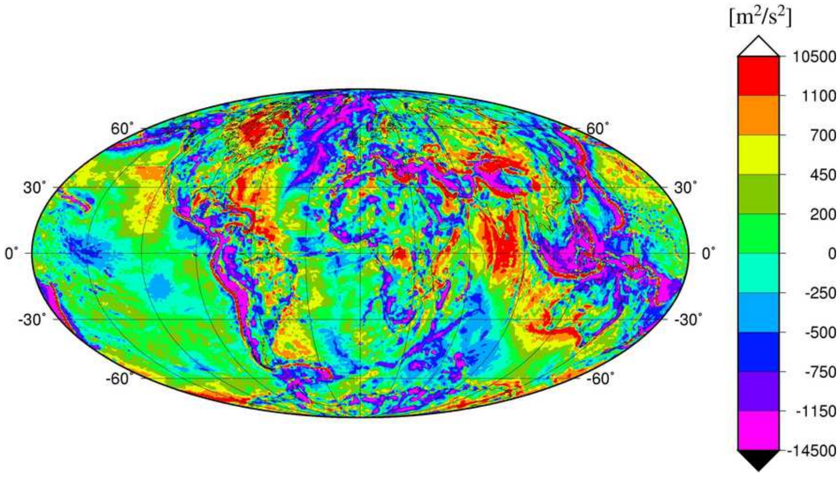


Figure 7.18: Boundary values.

As compared to Fig. 7.8, which provides the boundary information associated with the simple isotropic approach, it can be stated that the boundary data shown above are of considerably smaller magnitude. Moreover, a well-defined correlation with the topographical structure of the Earth can be observed in the case under investigation. These findings can be pointed out even better by plotting the basic quantities along the same Himalayan meridian as before in Fig. 7.11.

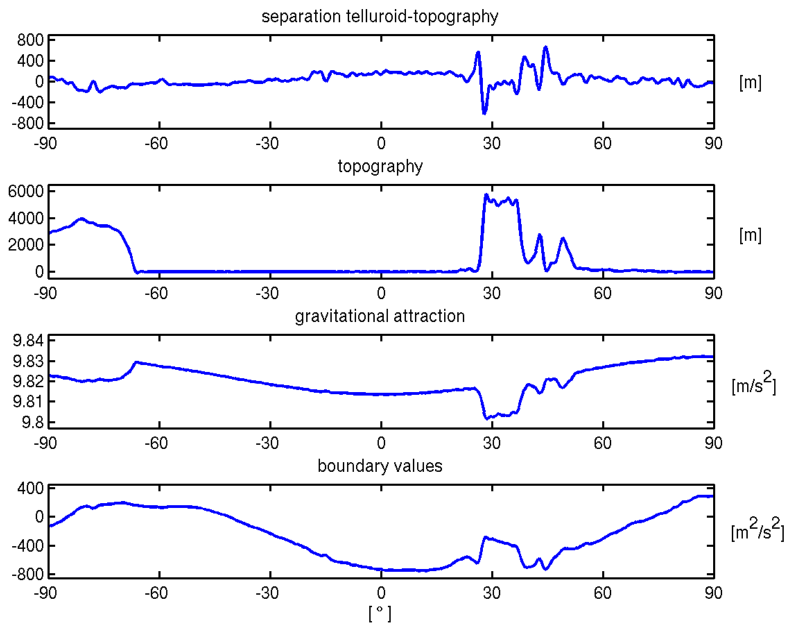


Figure 7.19: Profiles of the relevant quantities in the Himalayas.

In point of fact, it can be concluded that the boundary values associated with the presently considered approach are sized substantially smaller, to be exact a factor of about one hundred applies, and possess an enhanced sensitivity to the Earth's topography. This behavior can be identified as a direct consequence of the reduced separation between the boundary surface and the surface of the Earth.

7-2.4 Numerical BVP solution and closed-loop result

As far as the actual numerical solution of the corresponding BVP is concerned, it is simply referred to Section 7-1.4. Of course, instead of solving the BVP according to Definition 25, the solution is achieved for the BVP specified by Definition 33. Accordingly, the modified adjoint disturbing potential $\delta\bar{\psi}$, cf. (6.90) in Section 6-3, is determined in place of the former adjoint disturbing potential $\delta\psi$, see (5.82) in Section 5-3. Furthermore, in contrast to (7.30), a new height anomaly results

$$\bar{\boldsymbol{\zeta}} = \bar{\gamma}(\boldsymbol{\xi}\big|_{\Sigma})\nabla_{\xi}\delta\bar{\psi}(\boldsymbol{\xi}\big|_{\Sigma})\,. \tag{7.41}$$

In view of Lemma 39 introduced in Section 6-4, this relationship can even be reduced to

$$\bar{\boldsymbol{\zeta}} = \gamma(\boldsymbol{\xi}\big|_{\Sigma})\nabla_{\xi}\delta\bar{\psi}(\boldsymbol{\xi}\big|_{\Sigma})\,. \tag{7.42}$$

At length, the vector $\bar{\boldsymbol{\zeta}}$ is used again to correct for the discrepancies between the Earth's surface and the associated approximation surface. Once more, the latter is of the same geometrical structure as the corresponding boundary surface Σ in gravity space obtained from (7.38). Its realization in ordinary space is conducted by means of $\mathbf{x}\big|_{\Sigma}$. Hence, it applies

$$\mathbf{x}\big|_{\sigma} = \mathbf{x}\big|_{\Sigma} + \bar{\boldsymbol{\zeta}}\,. \tag{7.43}$$

Other than that, the overall way of proceeding remains the same. That is, similarly to (7.20) and (7.22), the adjoint disturbing potential $\delta\bar{\psi}$ and the boundary data $\Delta\bar{v}$ are represented in terms of spherical harmonics series expansions. In addition, especially the solution procedure remains the same, i.e. the parameter determination within the GSHA step and the subsequent comparison of coefficients in the spectral domain.

For this reason, the presentation of the final closed-loop result is given right away. Again, following the example given in the context of Fig. 7.12, the variations between the target topography and the topography recovered from (7.43) is exemplified next.

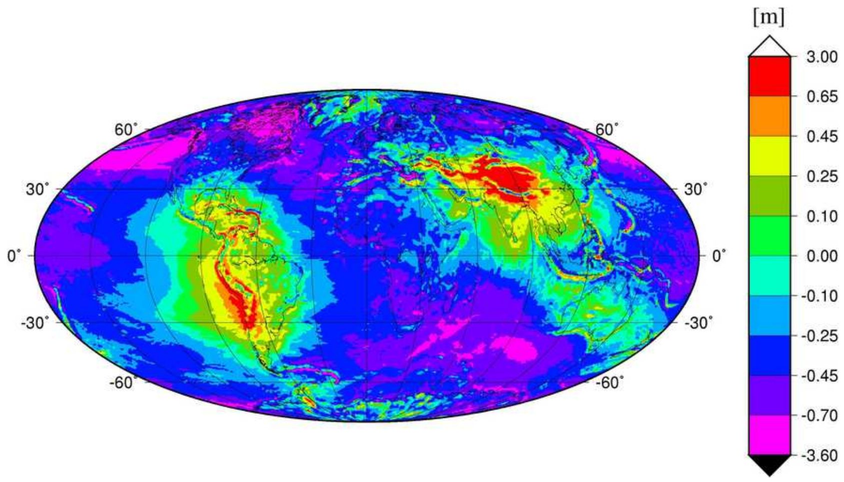
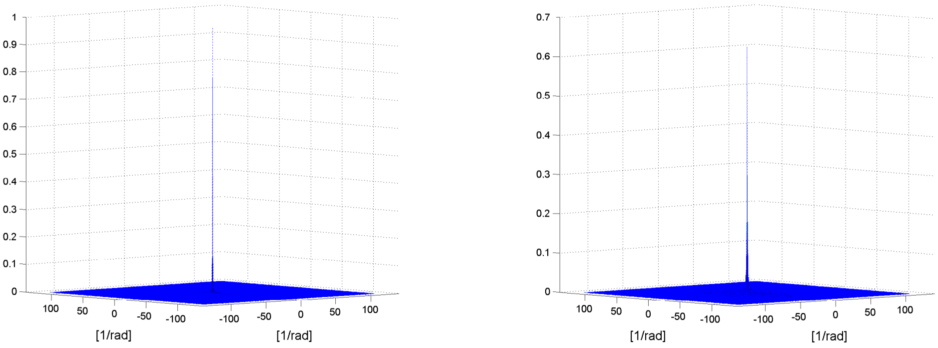


Figure 7.20: Closed-loop comparison of given and recovered topography.

The result is absolutely satisfying. The closed-loop error diminishes significantly from the meter level, cf. Tab. 7.1, down to the decimeter. A more exact quantification can be drawn from the Table 7.2 given on the next page. According to it, the mean error is about 0.4 m with peak errors below 4 m. Besides the crucial recovery of a reduced error level, the plot shown in Fig. 7.20 features at least two further important properties. On the one hand, an increased correlation of the illustrated error with the topographical structures can be testified. On the other hand, it seems reasonable to assume that the observed errors are more uniformly distributed. That is, no predominant offset to such an extent as has been pointed out previously within the scope of the examination of Fig 7.12 can

RMS [m]	mean [m]	std [m]	max [m]	min [m]
0.44	0.36	0.25	3.07	-3.79

Table 7.2: Statistics of closed-loop comparison of given and recovered topography.



(a) isotropic approach (b) ellipsoidal approach

Figure 7.21: 2d Fourier spectra of the closed-loop topography differences.

be identified. This reasoning is additionally supported by Fig. 7.21, where for the two regular approaches the 2d Fourier spectra of the closed-loop errors are compared.

In fact, the addressed findings can be related to the new simulation environment based on the ellipsoidal approach. As has been elaborated in Section 7-2.2, the revised linearization concept leads to an increased conformity of the boundary surface Σ and the surface of the Earth σ. As discussed in Section 7-2.2 and Section 7-2.3, the high conformity of Σ and σ involves a decrease in distortions between the data and the computation grid and leads to smaller-sized as well as to topography-sensitive boundary values. In addition, the absence of the error offset evidenced the last time indicates that in the present case the direct projection of the boundary values $\Delta\bar{v}$ is tolerable. Hence, dependent on the desired accuracy ,the necessity to apply explicit data continuation strategies, ideally supported by re-gridding methods, as has been considered advisable in the case of the isotropic approach, cf. Section 7-1.5, can probably be abandoned. All things considered, the final result is fundamentally satisfying, especially against the background of the fact that according to Section 7-2.1 the error level could still be improved to a certain extent by taking a higher degree series expansion for the underlying transformation formulae into account. Thus, the numerical proof of concept for the ellipsoidal regular gravity space approach has been accomplished with success.

7-3 Brief summary on the numerical results

In conclusion, both regular gravity space approaches were successfully implemented for the first time. It turned out that it has only been possible to achieve a general numerical proof of concept for the isotropic regular gravity space approach, whereas the ellipsoidal regular gravity space approach proved to be not only applicable in general but also to be numerically competitive as compared to the standard methods, such as Molodensky's treatment of the GBVP. Moreover, beyond competitiveness the revised regular gravity space approach exhibits a distinct difference to the traditional approach of Molodensky that should be highlighted once again. First of all, a remarkable similarity of the respective linearized BVPs has been worked out, cf. Definitions 12 and 33. In fact, both problems are of the same mathematical structure, only that potential and gravity have changed their places. That is, within the linearized Molodensky problem the boundary surface is defined by a potential-based relationship, see Definition 6, whereas within the linearized gravity space approach the boundary surface is defined by a gravity-based relationship, see Definition 28. On the other hand, gravity anomalies constitute the boundary data for the Molodensky problem, while for gravity space approach the boundary values are defined in terms of potential anomalies. As already pointed out in the context of Fig. 7.9 given in Section 7-1.3, the boundary data Δv within the gravity space approach are smoother than the boundary data $\Delta\Gamma$ in the case of Molodensky's problem. This is conceptually understandable

since both field quantities are measured at the Earth's surface but the potential is naturally smoother than its gradient. The fact that in the framework of the ellipsoidal regular gravity space approach the measured potential values are mapped to the gravimetric telluroid and, thereafter, to the computational sphere does not alter their smoothness. Furthermore, the potential data Δv are related to spirit-leveling and are therefore available with a higher density than the gravity measurement related gravity anomalies $\Delta\Gamma$ of the Molodensky problem. Even more, on the oceans, besides the small influence of the dynamical topography, the potential V is constant. Hence, the boundary values on the oceans are continuously available.
For all these reasons, it seems reasonable to draw the conclusion that the GBVP can also effectively be approached within the scope of the introduced modified regular gravity space approach. At last, together with a closing summary, all relevant advantages and disadvantages of the traditional Molodensky method and of the presented gravity space approaches are contrasted once more in the final chapter given next.

Chapter 8

Concluding remarks

8-1 Final summary

In the nineteen-eighties the gravity space formulation of F. Sansò to solve the GBVP found a strong echo. His approach transformed the GBVP, a problem with an unknown boundary surface, into a BVP of gravity space with a fixed boundary surface. This reduced the mathematical complexity substantially. Yet soon after, with the recovery of a singularity inherent in F. Sansò's approach, the method ceased to be widely discussed again without ever being really numerically implemented. Hence, the starting point of this thesis has been the idea that the existence of a singularity within the mathematical formulation of a true physical datum can only be attributed to an improperly posed mathematical model. As a result, it has been anticipated that a singularity-free formulation must exist, which can also be numerically implemented.

Actually, the fact that the Legendre type of contact transformation, which forms the basis of F. Sansò's gravity space transformation, maps a point in infinity into a point in the origin has been identified beyond doubt to be the cause of the existing singularity. More precisely, since the concept of differentiability has no meaning at infinity, a singularity occurs for the corresponding image of the point at infinity. Remedy was found in terms of a new contact transformation introduced by W. Keller, which leaves an infinite point in infinity. The resulting gravity space approach has been found to be free of any singularities. Consequently, the new approach according to W. Keller has been referred to as singularity-free or regular gravity space approach.

For the first time, W. Keller's regular gravity space approach has been numerically implemented. During the evaluation of the first numerical results, it turned out that the new regular gravity space concept did not account for the flattening of the Earth's body as well as for the flattening of the terrestrial potential. This shortcoming led to a rather unsatisfactory quality of the numerical results. Moreover, possible improvements deemed to be computationally intensive. Thus, the only true alternative has been to look for a revised regular gravity space transformation that also takes into account the effects due to the geometrical and physical flattening.

For this purpose, a peculiarity of the simple regular gravity space approach has been utilized, namely the fact that this regular gravity space transformation reduces to an identical transformation if the true potential is set equal to the isotropic normal potential. As a consequence thereof, the modified regular gravity space formulation has been established on the basis that the new transformation becomes the identity mapping if the actual potential coincides with an ellipsoidal reference potential. By means of the implicit function theorem it was possible to prove that such a transformation must exist in general. Even though an explicit representation of the transformation in terms of a closed form expression could not be achieved, a representation in terms of a series expansion with respect to the flattening parameter has been found. Despite the fact that the truncated series representation constitutes only an approximation of the true transformation, the numerical closed-loop results were highly satisfying.

To sum up, the main goals of the present study have been accomplished. In addition to a thorough review of the classical Molodensky theory and F. Sansò's ansatz, the following necessary mathematical investigations have been performed for the new regular gravity space formulations. The first concern has always been the setup of the transformation equations under the terms of the desired characteristics such as the non-existence of singularities or the identical mapping property. Next, the underlying nonlinear BVPs of regular gravity space have been established. Thereafter, the crucial linearization step to deduce the corresponding linear BVPs has been the center of attention. At last, further simplifications such as spherical and constant radius approximations yielded the required BVP representations suitable for the later numerical experiments. Considered together, the results achieved within the scope of the presented study can be used to close a theoretical gap in terms of existence and uniqueness in the theory of the GBVP. Furthermore, the general applicability of the regular gravity space approach by W. Keller and, in particular, of the proposed modified approach based on an ellipsoidal concept was successfully demonstrated. As far as the latter methodology is concerned, primarily the refinement in terms of an ellipsoidal formulation within the linearization step led to highly promising numerical results. Nevertheless, also a limited suitability of the regular gravity space approach based on an isotropic framework has been shown. Further numerical and conceptual improvements could probably also guarantee the practical usefulness of this approach.

8-2 Discussion

In detail, the following findings and innovations were achieved throughout this work. It could be proved that the solution of the GBVP by means of a gravity space based approach profits from two essential factors. For one thing, the overall problem transforms from a free into a fixed BVP, for another thing and despite the fact that the boundary surface gets rougher, smoother potential anomalies instead of gravity anomalies constitute the underlying boundary data. Furthermore, due to the identical mathematical structure of the involved linear BVPs, all algorithms and procedures developed for the simple Molodensky's problem can be carried over to the linearized regular gravity space approaches.

Hence, everything points to the fact that the solution of the GBVP by means of a regular gravity space formulation, in particular using the modified concept based on an ellipsoidal linearization point, is at least competitive to the classical solution approach of M.S. Molodensky. On the other hand, the consideration of either of the two regular gravity space approaches instead of the usual Molodensky problem to solve the GBVP exhibits one conceptual disadvantage, which must not be left unattended.

In geometry space two versions of Molodensky's problem are considered. That is, the so-called vectorial form of Molodensky's problem, where the complete gravity vector or rather vectorial gravity anomalies are used as boundary data, and the so-called scalar form of Molodensky's problem, where only the modulus of the aforementioned quantities enters the problem as boundary data. From the observational point of view the modulus of the gravity vector, i.e. gravity, is much easier to access than the full gravity vector. Moreover, astronomical observations are not precise enough to derive geodetic latitude and longitude. Hence, in geometry space the scalar form of Molodensky's problem is the prevailing form of dealing with the GBVP.

In contrast to this, the gravity space approach introduced by F. Sansò is an example of a so-called contact transformation. Contact transformations, first used by Sophus Lie and Felix Klein, are based on the idea to consider the solution-manifold of a differential equation as the envelope of its tangential spaces and to transform the equation for the manifold into an equivalent equation for its tangential spaces. The latter then is hopefully easier to treat than the original one. Since the quantity that describes the tangential plane of a manifold V is the gradient ∇V, the complete gradient and not only its modulus has to be known in order to apply a contact-transformation. Therefore a contact transformation or in other words a gravity space transformation for the scalar Molodensky problem cannot be found, despite the fact that the scalar formulation is much closer to reality.

Thus, in case the full gravity vector is not available as necessary but only its modulus, the missing horizontal information would have to be reconstructed from a known GPM. Naturally, this results in a horizontal uncertainty smaller than about 2 km for the boundary surface or, expressed as a relative deviation, of $2 \cdot 10^{-4}$. On the other hand, the boundary surface, except in the case of the boundary element method, will as a matter of fact seldom be used as a computation surface. Instead of that, all data given on the actual boundary surface will be continued or, as realized so far, projected to a bounding sphere. As elaborated before, the projection error related to the deviations between the boundary surface and the sphere or the error due to nodal point distortions equal or even exceed the abovementioned error level. This means, a relative error of $2 \cdot 10^{-4}$ related to a possible lack of knowledge in vertical deflections is tolerable, at least for the purpose presently intended.

Apart from such considerations of rather theoretical nature – bearing in mind that already the assumption of global data coverage is not applicable – the further development of F. Sansò's approach right up to the new ellipsoidal regular gravity space approach comprises the following specific amendments. Recapitulating in brief, the progress from F. Sansò's gravity space approach to W. Keller's regular gravity space concept eliminated the singularity immanent in the first gravity space formulation. The new regular transformation also led directly to the required harmonicity of the adjoint disturbing potential. In fact, pursuing further investigations on F. Sansò's approach was abandoned due to the underlying singularity plus the lack of harmonicity of the adjoint disturbing potential. As a result, a striking parallelism of the new regular approach and the simple Molodensky's problem could be made out, including the retrieval of the real physical image in terms of metric units and exterior BVP structure, two characteristics that have additionally been lost within the scope of F. Sansò's gravity space approach. Furthermore, whereas F. Sansò's methodology still involved the use of three relevant reference surfaces, W. Keller's revised ansatz gets along with only two boundary surfaces. The reason for this has been identified in terms of the so-called identity mapping, a property inherent in the regular gravity space approach of W. Keller. Apart from a slightly more complicated set of equations to transform between geometry and dual space, no further disadvantages of the regular gravity space approach with respect to the early gravity space approach of F. Sansò could be recovered.

Only the first computational studies on the regular gravity space approach revealed that the underlying isotropic linearization point leads to rather dissatisfying numerical results. A finding that most likely applies to F. Sansò's approach as well, since his procedure also implies an isotropic normal potential within the linearization step. As a consequence, a second regular gravity space approach based on the revised concept of an ellipsoidal linearization point has been introduced within the framework of this thesis. The elliptical modification of the isotropic regular gravity space approach resulted in a far better geometrical approximation of the Earth's surface by the corresponding boundary surface in gravity space. The reduced magnitude of the position anomalies or, in other words, the improved conformity of both surfaces was directly accompanied by smaller horizontal grid distortions and, together with the application of an ellipsoidal reference potential, by smaller-sized and more sensitive boundary values. On the other hand, the above mentioned higher conformity of the respective surfaces in geometry and dual space could only be guaranteed by an increase in ellipticity of the boundary surface in ellipsoidal regular gravity space. In turn, this resulted to larger separations between the boundary surface and the computational sphere as compared to the isotropic approach. In spite of it, the spheroidal regular gravity space approach overexceeds the isotropic gravity space approach in terms of the numerical performance by far, since the advantages in the form of reduced distortions and superior boundary values outweigh the depicted disadvantage. As a matter of fact, even without the use of explicit data continuation, the resulting closed-loop error of less than 0.4 m for spheroidal regular gravity space approach is entirely satisfying.
Admittedly, a single instance is, to a certain extent, limiting the theoretical beauty of the new ellipsoidal approach. That is to say, in order to come up with Laplace's equation as the field equation that must be satisfied by the adjoint disturbing potential, performing solely the linearization step as customary turned out to be insufficient. Indeed, to obtain Laplace's equation as the relevant field equation has only been possible within the subsequent spherical approximation step. More precisely, in contrast to the isotropic approach, where exclusively the boundary operator has to be simplified, the ellipsoidal approach also involves an approximation of the underlying field equation. Only then can the Molodensky-type of mathematical structure also be preserved for the BVP associated with the new ellipsoidal linearization strategy. Certainly, the numerical results accomplished under these circumstances demonstrate that this limitation is hardly worth mentioning.

All things considered, the radically new idea of F. Sansò to solve the GBVP in gravity space has already widely been considered to be a milestone in the theory of solving the fundamental problem of geodesy. Consequently, the current achievement of closing a gap in the theory of the gravity space approaches, which was still left open in terms of the singularity inherent in F. Sansò's approach, might be valued at least as highly.

8-3 Recommendations and outlook

It is in the nature of things that several open issues, especially of practical importance, come along with a thesis, which is essentially oriented to cultivate the basic principles related to the underlying theory of the GBVP. Hence, there are numerous chances and needs for future research.
In detail, the following improvements of conceptual nature are conceivable. First of all, the use of explicit data continuation, e.g. by means of collocation, could be envisaged. In addition, if required at all, a re-gridding strategy could be embedded in the data continuation process. Both measures aim at a refinement of the numerical performance of the GSHA step. Beyond it, an iterated solution method could be envisioned. In this case, the solution received for the disturbing potential or rather adjoint disturbing potential is used to correct the boundary data for the corresponding influence at the end of the first iteration step. In a second step, the solution process is restarted based on the reduced boundary information.

Alternatively, besides such methodical considerations, extending the numerical procedures can also contribute to increase the overall accuracy. For example, the application of a higher order series expansion for the representation of the transformation formula within the ellipsoidal regular gravity space approach might be a simple measure to further minimize the closed-loop error. Moreover, instead of taking the recommended explicit data continuation step into account, substitution of the former spherical harmonic analysis procedure by means of a more appropriate ellipsoidal harmonic analysis procedure might be a worthwhile alternative. Naturally, this would involve the application of the much-needed elliptically-shaped computational surface that, in contrast to the sphere, optimally complies with the boundary surfaces of geometry and gravity space.

One the other hand, apart from the above quoted aspects focusing on the upgrade of the achievable accuracy with the global scenario, the adaption of the presented methods to regional and local situations should be advanced. In this context, as opposed to the prevailing three-dimensional consideration, a planar approximation approach could come into question.
At last, the idea of navigating in gravity space might be an interesting and promising possibility. As has been demonstrated, the determination of the geometrical figure of the Earth from physical quantities such as potential and potential gradient is state-of-the-art. From now onwards, on the basis of gravity space theory, even an arbitrary geometrical position outside the Earth's masses can be deduced from the corresponding gravity vector and the knowledge of the spatial behavior of the adjoint potential.

Appendix A

Legendre functions

A wealth of literature is devoted to the theory of spherical harmonics, e.g. [30] Hobson 1931. This implicitly involves the subject-matter of evaluating the closely related Legendre-functions, see also Section 2-2.4. For the latter aspect, relevant literature would be, e.g., [73] Paul 1978; [37] Ilk 1983; [100] Wenzel 1985; [98] Thong 1989; [38],[39] Jekeli 1996,2007 and [107] Wittwer 2008. The relations presented in the following are compiled from these contributions.

To begin with, the definition of Ferrer's associated Legendre-functions of 1st kind reads as follows

$$P_{kl}(t) = \frac{1}{2^k k!}(1-t^2)^{l/2}\frac{d^{k+l}}{dt^{k+l}}(t^2-1)^k \quad ; \qquad k \geq l \geq 0\,, \tag{A.1}$$

which is subject to the argument

$$t = \sin\phi = \cos\vartheta \quad ; \qquad -1 \leq t \leq 1 \;; \quad -\pi/2 \leq \phi \leq \pi/2 \;; \quad 0 \leq \vartheta \leq \pi\,. \tag{A.2}$$

Additionally, it holds

$$P_{k,-l}(t) = (-1)^l \frac{(k-l)!}{(k+l)!} P_{kl}(t) \tag{A.3}$$

$$P_{kl}(-t) = (-1)^{k+l} P_{kl}(t)\,. \tag{A.4}$$

Ferrer's fully normalized associated Legendre-functions result from

$$P^*_{kl}(t) = n_{kl} P_{kl}(t) \quad ; \qquad k \geq l \geq 0\,, \tag{A.5}$$

which is subject to the following normalization coefficient

$$n_{kl} := \sqrt{(2-\delta_{l0})(2k+1)\frac{(k-l)!}{(k+l)!}} \tag{A.6}$$

and

$$\delta_{l0} := \begin{cases} 1 & ; \quad l = 0 \\ 0 & ; \quad l \neq 0 \end{cases}$$

Furthermore, it applies

$$P^*_{k,-l}(t) = (-1)^l P^*_{kl}(t)\,. \tag{A.7}$$

Subsequent to the above considerations defining the associated Legendre functions of 1st kind in normalized and non-normalized form, an overview on the available formulae for the recursive computation of these functions is given. At first, one amongst many sets of recurrence relations to deploy the associated Legendre functions of 1st kind is supplied

$$\begin{array}{lrcll}
a) & P_{0,0} & = & 1.0 & \text{(starting value)} \\
b) & P_{k+1,k+1}(t) & = & (2k+1)\sqrt{1-t^2}P_{k,k}(t) & ; \quad k \geq 0 \\
c) & P_{k+1,k}(t) & = & (2k+1)tP_{k,k}(t) & ; \quad k \geq 0 \\
d) & P_{k+1,l}(t) & = & \dfrac{1}{(k-l+1)}[(2k+1)tP_{k,l}(t) - (k+l)P_{k-1,l}(t)] & ; \quad k \geq (l+1)\ ,\ l \geq 0\,.
\end{array} \tag{A.8}$$

Secondly, the direct recursive computation of the fully normalized Legendre functions of 1st kind can be achieved according to

a) starting values

$$\begin{aligned} P^*_{0,0}(t) &= 1.0 \\ P^*_{1,0}(t) &= \sqrt{3}t \\ P^*_{1,1}(t) &= \sqrt{3}\sqrt{1-t^2} \\ P^*_{2,0}(t) &= \frac{\sqrt{5}}{2}(3t^2-1) \\ P^*_{2,1}(t) &= \sqrt{15}t\sqrt{1-t^2} \\ P^*_{2,2}(t) &= \frac{\sqrt{15}}{2}(1-t^2) \end{aligned} \tag{A.9}$$

and

$$\begin{aligned} b)\quad P^*_{k,k-1}(t) &= \sqrt{\frac{2k+1}{2(k-1)}}\sqrt{1-t^2}P^*_{k-1,k-2} \quad ; \quad k>2 \\ c)\quad P^*_{k,k}(t) &= \sqrt{\frac{2k+1}{2k}}\sqrt{1-t^2}P^*_{k-1,k-1} \quad ; \quad k>2 \\ d)\quad P^*_{k,l}(t) &= \sqrt{\frac{4k^2-1}{k^2-l^2}}tP^*_{k-1,l} - \sqrt{\frac{(2k+1)((k-1)^2-l^2)}{(2k-3)(k^2-l^2)}}P^*_{k-2,l} \quad ; \; k>2\,,\; 0\leq l\leq k-2\,. \end{aligned} \tag{A.10}$$

For example, for the purpose of evaluating (2.65), given in Section 2-2.4, the necessity to compute additionally the first order derivatives of the prior introduced Legendre functions kind can be understood. Hence, the relationship to determine the first order derivatives of the associated Legendre functions of the 1st kind is provided in the first place. In this context, two peculiarities need special notification. On the one hand, the poles must be excluded from the solution domain and, on the other hand, the argument t has to be explicitly declared due to the chain rule of differential calculus. Consequently, on choosing $t=\sin\phi$ it applies

$$P_{k,l}{}'(\sin\phi) = \frac{1}{\cos\phi}\left[(k+1)\sin\phi P_{k,l}(\sin\phi) - (k-l+1)P_{k+1,l}(\sin\phi)\right] \quad ; \qquad k\geq l\geq 0\,; \quad -1<t<1\,. \tag{A.11}$$

At last, the first order derivatives of the fully normalized Legendre functions of the 1st kind result from

$$\begin{aligned} a)\quad P^*_{0,0}{}'(\sin\phi) &= 0 \qquad \text{(starting value)} \\ b)\quad P^*_{k,0}{}'(\sin\phi) &= \sqrt{\frac{k(k+1)}{2}}P^*_{k,1} \quad ; \; k>0 \\ c)\quad P^*_{k,1}{}'(\sin\phi) &= -\sqrt{\frac{k(k+1)}{2}}P^*_{k,0} + \frac{\sqrt{k(k+1)-2}}{2}P^*_{k,2} \quad ; \; k>0 \\ d)\quad P^*_{k,k}{}'(\sin\phi) &= -\sqrt{\frac{k}{2}}P^*_{k,k-1} \quad ; \; k>1 \\ e)\quad P^*_{k,l}{}'(\sin\phi) &= -\frac{\sqrt{k(k+1)-l(l-1)}}{2}P^*_{k,l-1} + \frac{\sqrt{k(k+1)-l(l+1)}}{2}P^*_{k,l+1} \quad ; \; k>1\;,\; 2\leq l\leq k-1\,. \end{aligned} \tag{A.12}$$

Interestingly, with a view to (A.11) and (A.12), it is worth pointing out that the computation of the derivatives of the Legendre functions according to these relations is based solely on the use of the Legendre functions themselves. On the other hand, the formulae given, e.g., in [100] Wenzel 1985 also utilize previously derived values within the recurrence.

Appendix B

Supplements to the ellipsoidal regular gravity space approach

B-1 The auxiliary vector q

As previously stated in connection with (6.73) and Remark 27 given in Section 6-1.3, the vector $\mathbf{q}(\boldsymbol{\xi})$, constituted in terms of a series representation with respect to the small parameter J_2, has been explicitly derived using a computer algebra system (CAS), such as e.g. *Mathematica*. The resulting expression achieved for the vector $\mathbf{q}(\boldsymbol{\xi})$ is obtained stepwise by means of the following Mathematica commands given in the form of the actual programming code:

CAS Statement 1 :
```
<< Calculus`VectorAnalysis`
SetCoordinates[Cartesian[x, y, z]]
```

CAS Statement 2 : $\mathtt{V} = \frac{\mathtt{GM}}{\sqrt{\mathtt{x}^2+\mathtt{y}^2+\mathtt{z}^2}}\left(1 - \mathtt{J}_2\left(\frac{\mathtt{R}^2}{\mathtt{x}^2+\mathtt{y}^2+\mathtt{z}^2}\right)\mathtt{LegendreP}\left[2, \frac{\mathtt{z}}{\sqrt{\mathtt{x}^2+\mathtt{y}^2+\mathtt{z}^2}}\right]\right)$

CAS Statement 3 : `p=Simplify[Grad[V]]`

CAS Statement 4 : `matV=Simplify[Grad[p]]`

CAS Statement 5 : `invV=Simplify[Inverse[matV]]`

CAS Statement 6 : `prod=Simplify[p.invV]`

CAS Statement 7 : `q=Simplify[Series[prod, {J₂,0,2}]]`

$$
\mathtt{Out[7]} = \begin{array}{l}
-\frac{x}{2} + \frac{3R^2x(x^2+y^2+2z^2)J_2}{4(x^2+y^2+z^2)^2} - \frac{9(R^4x(x^4+y^4-2y^2z^2-12z^4+2x^2(y^2-z^2)))J_2^2}{4(x^2+y^2+z^2)^4} + \mathrm{O}\left(J_2^3\right) \\
-\frac{y}{2} + \frac{3R^2y(x^2+y^2+2z^2)J_2}{4(x^2+y^2+z^2)^2} - \frac{9(R^4y(x^4+y^4-2y^2z^2-12z^4+2x^2(y^2-z^2)))J_2^2}{4(x^2+y^2+z^2)^4} + \mathrm{O}\left(J_2^3\right) \\
-\frac{z}{2} - \frac{3(R^2z(3x^2+3y^2+2z^2))J_2}{4(x^2+y^2+z^2)^2} + \frac{9R^4z(9x^4+9y^4-4y^2z^2-4z^4+2x^2(9y^2-2z^2))J_2^2}{4(x^2+y^2+z^2)^4} + \mathrm{O}\left(J_2^3\right)
\end{array}
\qquad (B.1)
$$

B-2 Derivation of the vector $\xi|_\Sigma$

In the context of deriving the position anomaly vector $\bar{\boldsymbol{\zeta}}$ at the end of Section 6-3, the assumption was made that it can be shown by means of utilizing a CAS, such as Mathematica, that

$$\left(\bar{\gamma}\nabla_\xi \bar{\psi}_0\right)|_\Sigma = \boldsymbol{\xi}|_\Sigma$$

holds true, cf. (6.105). Again, the relevant Mathematica commands given in terms of the actual programming code are provided to verify this identity:

CAS Statement 1 :
```
<< Calculus`VectorAnalysis`
SetCoordinates[Cartesian[x, y, z]]
```

CAS Statement 2 : $\mathtt{V} = \frac{\mathtt{GM}}{\sqrt{\mathtt{x}^2+\mathtt{y}^2+\mathtt{z}^2}}\left(1 - \mathtt{J}_2\left(\frac{\mathtt{R}^2}{\mathtt{x}^2+\mathtt{y}^2+\mathtt{z}^2}\right)\mathtt{LegendreP}\left[2, \frac{\mathtt{z}}{\sqrt{\mathtt{x}^2+\mathtt{y}^2+\mathtt{z}^2}}\right]\right)$

CAS Statement 3 :
```
p=Simplify[Grad[V]]
```

CAS Statement 4 :
```
vecX={x,y,z}
```

CAS Statement 5 :
```
psi0=Simplify[p.vecX-V]
```

Out[5] = $-\frac{2\mathtt{GM}\left((\mathtt{x}^2+\mathtt{y}^2+\mathtt{z}^2)^2+\mathtt{R}^2(\mathtt{x}^2+\mathtt{y}^2-2\mathtt{z}^2)\mathtt{J}_2\right)}{(\mathtt{x}^2+\mathtt{y}^2+\mathtt{z}^2)^{5/2}}$

CAS Statement 6 :
```
alphabar=Simplify[Grad[p]]
```

CAS Statement 7 :
```
gammabar=Simplify[Inverse[alphabar]]
```

CAS Statement 8 :
```
pi0=Simplify[Grad[psi0]]
```

CAS Statement 9 :
```
gammapi0=Simplify[gammabar.pi0]
```

Out[9] = {x,y,z} ◊

Appendix C

Numerical methods

C-1 Newton's method

For the purpose of the point-wise determination of the boundary surface Σ in the context of Section 7-1.2, the numerical evaluation of (7.5), i.e.

$$\nabla V_0(\boldsymbol{\xi}_{ij}\big|_\Sigma) = \tilde{\mathbf{g}}_{ij} = \nabla V(\mathbf{x}_{ij}\big|_\sigma)\,,$$

by means of Newton's method has been taken into consideration as a testing alternative for the usual transformation approach (7.4). Hence, the computation of the gravimetric telluroid Σ, see Definition 21 given in Section 5-1.2, takes place in an iterative manner by equating the gradient of the normal potential at the gravimetric telluroid Σ to the gradient of the true potential at the Earth's surface σ, cf. Lemma 17 also given in Section 5-1.2. In detail, the application of Newton's method to the case under consideration is as follows. The iteration in terms of $\boldsymbol{\xi}_{ij}^n$, defining the required telluroid points, is started at the corresponding topography points $\mathbf{x}_{ij}\big|_\sigma$ and proceeds according to Newton's familiar recurrence relation

$$\begin{aligned}
\boldsymbol{\xi}_{ij}^0\big|_\Sigma &= \mathbf{x}_{ij}\big|_\sigma \\
\boldsymbol{\xi}_{ij}^{n+1}\big|_\Sigma &= \boldsymbol{\xi}_{ij}^n\big|_\Sigma - \left[\nabla\left(\nabla V_0(\boldsymbol{\xi}_{ij}^n\big|_\Sigma)\right)\right]^{-1}\left(\nabla V_0(\boldsymbol{\xi}_{ij}^n\big|_\Sigma) - \nabla V(\mathbf{x}_{ij}\big|_\sigma)\right) \\
&= \boldsymbol{\xi}_{ij}^n\big|_\Sigma - \mathbf{V}_0^{-1}(\boldsymbol{\xi}_{ij}^n\big|_\Sigma)\left(\nabla V_0(\boldsymbol{\xi}_{ij}^n\big|_\Sigma) - \tilde{\mathbf{g}}_{ij}\right)\,,
\end{aligned}$$

which is subject to the Hessian matrix

$$\mathbf{V}_0^{-1}(\boldsymbol{\xi}) = \left[V_{kl}^0\right]^{-1} = -\frac{\|\boldsymbol{\xi}\|^3}{GM}\left[\delta_{kl} - \frac{3}{2}\frac{\xi_k\xi_l}{\|\boldsymbol{\xi}\|^2}\right]\,.$$

As far as the necessary number of iterations is concerned, it can be stated that $n = 5$ iterations are more than sufficient to achieve a sub-millimeter accuracy for the gravimetric telluroid.

C-2 Gauss-Legendre quadrature

The contributions [89],[90] Sneeuw 1993,1994 and [29] Hirsch 1996 of N. Sneeuw and M. Hirsch essentially provide the background of the algorithm applied throughout this work for the GSHA step. Within his publications, N. Sneeuw also nicely places the general subject-matter of global spherical harmonic computation (GSHC) in a historical context. In fact, he traces the beginning of GSHC, in particular of the currently considered two-step GSHC formalism, back to mathematicians like F. Neumann, cf. [69] Neumann 1838, and C.F. Gauss, cf. [13] Gauss 1839. Accordingly, the GSHA method under investigation is also based on a two-step formulation. In short, it can be outlined as follows. The first step consists of a Fourier transformation along parallels. The Fast Fourier Transform (FFT) is feasible since within the transition from continuous to discretized data the orthogonality of the base functions is preserved. In detail, the trigonometric base functions sine and cosine maintain their orthogonality property as long as the longitudinal gridding is equiangular, i.e. $\lambda_i = i\cdot\Delta\lambda$; $i = 0, 1, \ldots, 2K-1$; $\Delta\lambda = \frac{\pi}{K}$, as already discussed in Section 7-1.1. The second step is devoted to the computation of the spherical harmonic coefficients from the obtained Fourier coefficients. However, due to the loss of orthogonality of the Legendre functions at discrete points, complications result in the latitude direction. To overcome this latitudinal orthogonality problem, F. Neumann devised an exact quadrature method, referred to as *second Neumann method*,

Gaussian quadrature or *Gauss-Legendre quadrature*, which is based on certain quadrature weights to preserve orthogonality of the Legendre functions in the discrete case. As a consequence thereof, the evaluation points are fixed but not uniformly distributed. Strictly speaking, as mentioned before, they must be positioned in the roots of the Legendre polynomial $P_{K+1}(\sin\phi)$. Note, that this so-called *Gauss-grid*, as a matter of fact, deviates only slightly from the equiangular distribution given along the parallels. The quadrature weights and the roots of the Legendre polynomial are tabulated for various values of K, see e.g. [56] Krylov 1962 or [95] Stroud 1966.
In the following, the formulae associated with the prior considerations are given. Once more, the harmonic analysis on the sphere S of the boundary values Δv_{ij}, cf. (7.22), by means of the outlined combination of FFT and Gauss-Legendre quadrature, aims at the determination of the corresponding spherical harmonic coefficients. For the sake of convenience, the co-latitude θ is used instead of the latitude ϕ and the identifiers ij, indicating that in fact discrete samples must be considered, are omitted. Moreover, in contrast to (7.22), the spherical harmonic coefficients are denoted by $\bar{v}_{kl}$ and not by $\bar{c}_{kl}$. Hence,

$$\Delta v(\theta,\lambda) = \sum_{k=0}^{\infty}\sum_{l=-k}^{k} \bar{v}_{kl}\bar{Y}_{kl}(\theta,\lambda) \tag{C.1}$$

applies for boundary values. As is known, the determination of the required coefficients $\bar{v}_{kl}$ can be achieved by representing the definition of the coefficients $\bar{v}_{kl}$, i.e.

$$\bar{v}_{kl} = \frac{1}{4\pi R^2}\iint\limits_S \Delta v \bar{Y}_{kl}\, dS\,, \tag{C.2}$$

as an iterated integral

$$\begin{aligned}\bar{v}_{kl} &= \frac{1}{4\pi R^2}\int_0^{\pi} P^*_{kl}(\cos\theta)\int_0^{2\pi}\Delta v(\theta,\lambda)\left\{\begin{array}{c}\cos l\lambda\\ \sin l\lambda\end{array}\right\} R^2\sin\theta d\lambda d\theta\\ &= \frac{1}{4\pi}\int_0^{\pi}\sin\theta P^*_{kl}(\cos\theta)\int_0^{2\pi}\Delta v(\theta,\lambda)\left\{\begin{array}{c}\cos l\lambda\\ \sin l\lambda\end{array}\right\} d\lambda d\theta\,.\end{aligned} \tag{C.3}$$

The inner integral

$$\left\{\begin{array}{c}a_l(\theta)\\ b_l(\theta)\end{array}\right\} = \int_0^{2\pi}\Delta v(\theta,\lambda)\left\{\begin{array}{c}\cos l\lambda\\ \sin l\lambda\end{array}\right\} d\lambda \tag{C.4}$$

is evaluated by the trapezian quadrature rule as follows

$$\left\{\begin{array}{c}a_l(\theta)\\ b_l(\theta)\end{array}\right\} = \frac{1}{N}\sum_{i=1}^{2K}\Delta v(\theta,\lambda_i)\left\{\begin{array}{c}\cos l\lambda_i\\ \sin l\lambda_i\end{array}\right\} \tag{C.5}$$

subject to $\lambda_i = i\frac{\pi}{k}$ and $l = 0\ldots K$. A closer look to the equation (C.5) shows that the quantities $a_l(\theta)$ and $b_l(\theta)$ are exactly the discrete Fourier coefficients of $\Delta v(\theta,\bullet)$. This means instead of using the trapezian rule these quantities can be computed per parallel by the Fourier Transform. Because of the orthogonality properties of the sine and cosine functions and because of the periodicity of the functions on each parallel, the forward and backward Fourier transformations can more efficiently be performed by FFT as discussed in the papers of N. Sneeuw.

The remaining outer integral

$$\bar{v}_{kl} = \frac{1}{4\pi}\int_0^{\pi}\sin\theta P^*_{kl}(\cos\theta)\left\{\begin{array}{c}a_l(\theta)\\ b_l(\theta)\end{array}\right\} d\theta \tag{C.6}$$

can be transformed to

$$\bar{v}_{kl} = \frac{1}{4\pi}\int_{-1}^{1} P^*_{kl}(x)\left\{\begin{array}{c}a_l(\arccos x)\\ b_l(\arccos x)\end{array}\right\} dx \tag{C.7}$$

and evaluated by a Gaussian quadrature formula

$$\bar{v}_{kl} = \frac{1}{4\pi}\sum_{j=0}^{N/2} w_j P^*_{kl}(x_j)\left\{\begin{array}{c}a_l(\arccos x_j)\\ b_l(\arccos x_j)\end{array}\right\} \tag{C.8}$$

with w_j and x_j being the aforementioned weights and nodes of the Gauss-Legendre quadrature formula. The $(i\cdot\Delta\lambda, \phi_j = \arcsin x_j)$-grid has been chosen in such a way that the spherical harmonic analysis is exact up to degree and order $K = 767$, see also Section 7-1.1. As far as a possible test procedure for the correct numerical implementation of the GSHA algorithm is concerned, e.g. [29] Hirsch 1996 can be taken into account.

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