A note on the wedge reversion antisymmetry operation and 51 types of physical quantities in arbitrary dimensions

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The paper by Gopalan [(2020). *Acta Cryst.* A**76**, 318–327] presented an enumeration of the 41 physical quantity types in non-relativistic physics, in arbitrary dimensions, based on the formalism of Clifford algebra. Gopalan considered three antisymmetries: spatial inversion, $\bar{1}$, time reversal, $1'$, and wedge reversion, $1^{\dagger}$. A consideration of the set of all seven antisymmetries ($\bar{1}$, $1'$, $1^{\dagger}$, $1'^{\dagger}$, $\bar{1}^{\dagger}$, $\bar{1}'$, $\bar{1}'^{\dagger}$) leads to an extension of the results obtained by Gopalan. It is shown that there are 51 types of physical quantities with distinct symmetry properties in total.

The paper by Gopalan (2020) presented an enumeration of the 41 types of physical quantities in non-relativistic physics in arbitrary dimensions within the formalism of Clifford algebra (Lounesto, 2009). This classification is based on three anti-symmetries: spatial inversion, $\bar{1}$, time reversal, $1'$, and wedge reversion, $1^{\dagger}$. [Note that, in Clifford algebra, spatial inversion $\bar{1}$ is termed (main) grade involution (Lounesto, 2009).] The 41 types of multivectors representing physical quantities were derived and presented in Table 1 of Gopalan (2020). Gopalan's classification is an extension of the classification of three-dimensional vector-like physical quantities (Hlinka, 2014) to arbitrary dimensions.

The transformation of the physical properties represented by the principal multivectors S′, S, V′, V, B′, B, T′, T, or their combinations, under the antisymmetries $\bar{1}$, $1'$, $1^{\dagger}$ were considered by Gopalan (2020). S, V, B and T denote scalar, vector, bivector and trivector, respectively. The prime symbol ′ means invariance to time reversal, $1'$. There are three different outcomes of the operation, even (*e*), odd (*o*) or mixed (*m*). Mixed means that it is neither even nor odd, as explained in the short example in Table 1.

These outcomes for all physical properties are shown for each multivector type in Table 1 of Gopalan (2020) in columns titled 'Action of $\bar{1}$, $1'$, $1^{\dagger}$'. The actions of the remaining anti-symmetries $\bar{1}'$, $\bar{1}^{\dagger}$, $1'^{\dagger}$ and $\bar{1}'^{\dagger}$ were not given in Table 1 of Gopalan (2020). [In Clifford algebra, the product of spatial inversion and wedge reversion, $\bar{1}^{\dagger}$, is termed Clifford conjugation (Lounesto, 2009).] The consideration of all seven operations leads to new results, which are given here.

When a physical quantity is represented by a sum of two or more different principal multivector types then the action of at least four antisymmetries on this quantity gives mixed results. The analysis of the action of only three antisymmetries by Gopalan (2020) does not provide a unique solution.

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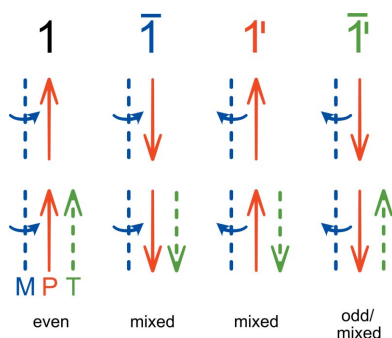
Table 1
An example of the action of $\bar{1}$, $1'$ and $\bar{1}'$ antisymmetries on several multivectors.

The action of antisymmetries $\bar{1}$ and $1'$ on S+V′ and S+V′+V gives mixed results, while the action of the product antisymmetry $\bar{1}'$ can be odd or mixed.

	$\bar{1}$	$1'$	$\bar{1}'$
S	+ Even	− Odd	− Odd
V′	− Odd	+ Even	− Odd
V	− Odd	− Odd	+ Even
S+V′	+− Mixed	−+ Mixed	−− Odd
S+V′+V	+−− Mixed	−+− Mixed	−−+ Mixed

Let I_1 and I_2 be two different antisymmetries (any out of the seven). If the action of both I_1 and I_2 on some multivector is mixed then the action of $I_1 \cdot I_2$ on this multivector can be even, odd or mixed. Specifying the action of only three antisymmetries (especially $\bar{1}$, $1'$ and $1^\dagger$) on a multivector, as was considered by Gopalan (2020), is not sufficient to obtain the result of the action of the remaining four antisymmetries; see a simplified example with three antisymmetries in Table 1. This has led to a clustering of different multivector types into one type in Table 1 of Gopalan (2020): the types numbered 16, 19, 22, 25, 28 and 31 should be separated into two types each and type 38 into five types. This gives in total ten new multivector types which were not given by Gopalan (2020), as shown in Table 2 for all seven antisymmetries. New labels for the X, Y, Z multivectors are proposed in Table 2 in a coherent notation, which uses four out of the eight principal multivectors. An extended version of Table 2 with grades and examples of multivectors is given in Table 3, which is the final table for these new results, with all 51 multivector types describing the action of all seven antisymmetries, given in the same layout as Table 1 of Gopalan (2020).

Acknowledgements

References

Gopalan, V. (2020). *Acta Cryst.* A**76**, 318–327.
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Lounesto, P. (2009). *Clifford Algebras and Spinors*, 2nd ed. Cambridge University Press.

Table 2
Splitting of multivector types, with the left-hand side displaying the number, stabilizer subgroup, label and action of $\bar{1}$, $1'$ and $1^\dagger$ antisymmetries as given by Gopalan (2020), and the right-hand side displaying the number, stabilizer subgroup, label and action of all antisymmetries as presented in this work.

Considering the action of all antisymmetries leads to splitting of multivector types. The last three rows describe the new labels of the X, Y and Z multivector types, without splitting. An extended version of this table with grades and examples of multivectors is available in the supporting information.

Work of Gopalan (2020)			Action of				This paper			Action of						
No.	SS	Label	$\bar{1}$	$1'$	$1^\dagger$		No.	SS	Label	$\bar{1}$	$1'$	$1^\dagger$	$1'^\dagger$	$\bar{1}'$	$\bar{1}^\dagger$	$\bar{1}'^\dagger$
16	$\bar{1}$	SB′(S′, B)	e	m	m	{	16a	$\bar{1}$	SB′	e	m	m	o	m	m	o
							16b		S′SB′B	e	m	m	m	m	m	m
19	$1'$	V′B′(S′, T′)	m	e	m	{	19a	$1'$	V′B′	m	e	m	m	m	o	o
							19b		S′V′B′T′	m	e	m	m	m	m	m
22	$1^\dagger$	SV′(S′, V)	m	m	e	{	22a	$1^\dagger$	SV′	m	m	e	m	o	m	o
							22b		S′SV′V	m	m	e	m	m	m	m
25	$1'^\dagger$	V′B(S′, T)	m	m	m	{	25a	$1'^\dagger$	V′B	m	m	m	e	o	o	m
							25b		S′V′BT	m	m	m	e	m	m	m
28	$\bar{1}'$	VB′(S′, T)	m	m	m	{	28a	$\bar{1}'$	VB′	m	m	m	o	e	o	m
							28b		S′VB′T	m	m	m	m	e	m	m
31	$\bar{1}^\dagger$	ST′(S′, T)	m	m	m	{	31a	$\bar{1}^\dagger$	ST′	m	m	m	o	o	e	m
							31b		S′ST′T	m	m	m	m	m	e	m
38	1	W	m	m	m	{	38a	1	SVB′T′	m	m	m	o	m	m	m
							38b		SV′BT′	m	m	m	m	o	m	m
							38c		V′VB′B	m	m	m	m	m	o	m
							38d		SV′B′T	m	m	m	m	m	m	o
							38e		S′SV′VB′BT′T	m	m	m	m	m	m	m
39	1	X	m	m	o	→	39	1	B′BT′T	m	m	o	m	m	m	m
40	1	Y	m	o	m	→	40	1	SVBT	m	o	m	m	m	m	m
41	1	Z	o	m	m	→	41	1	V′VT′T	o	m	m	m	m	m	m

Table 3
Classification of extended multivector types for physical properties using the same notation as in Table 1 of Gopalan (2020).

The actions of all seven generalized inversions and grades are given explicitly. Entries in bold in columns 1 and 2 are the eight principal multivector types. Note that the last column contains sums (not products) of principal multivectors, but the '+' signs are omitted.

New No.	Old No.	SS	Label	$\bar{1}$	$1'$	$1^\dagger$	$1'^\dagger$	$\bar{1}'$	$\bar{1}^\dagger$	$\bar{1}'^\dagger$	Grades	Multivectors (omitting '+' signs)
1	**1**	$\bar{1}1'1^\dagger$	S′	e	e	e	e	e	e	e	4g	S′
2	**2**	$1'1^\dagger$	V′	o	e	e	e	o	o	o	4g+1	V′
3	3		S′V′	m	e	e	e	m	m	m	4g, 4g′+1	S′V′
4	**4**	$\bar{1}1'$	B′	e	e	o	o	e	o	o	4g+2	B′
5	5		S′B′	e	e	m	m	e	m	m	4g, 4g′+2	S′B′
6	**6**	$\bar{1}^\dagger 1'$	T′	o	e	o	o	o	e	e	4g+3	T′
7	7		S′T′	m	e	m	m	m	e	e	4g, 4g′+3	S′T′
8	**8**	$\bar{1}1^\dagger$	S	e	o	e	o	o	e	o	4g	S
9	9		S′S	e	m	e	m	m	e	m	4g, 4g′	S′S
10	**10**	$\bar{1}'1^\dagger$	V	o	o	e	o	e	o	e	4g+1	V
11	11		S′V	m	m	e	m	e	m	e	4g, 4g′+1	S′V
12	**12**	$\bar{1}1'^\dagger$	B	e	o	o	e	o	o	e	4g+2	B
13	13		S′B	e	m	m	e	m	m	e	4g, 4g′+2	S′B
14	**14**	$\bar{1}^\dagger 1'^\dagger$	T	o	o	o	e	e	e	o	4g+3	T
15	15		S′T	m	m	m	e	e	e	m	4g, 4g′+3	S′T
16	16a		SB′	e	m	m	o	m	m	o	4g, 4g′+2	SB′
17	17	$\bar{1}$	SB	e	o	m	m	o	m	m	4g, 4g′+2	SB
18	18		B′B	e	m	o	m	m	o	m	4g+2, 4g′+2	B′B
19	16b		S′SB′B	e	m	m	m	m	m	m	Three or four out of: 4g, 4g′, 4g″+2, 4g‴+2	SB′B, S′B′B, S′SB, S′SB′, S′SB′B
20	19a		V′B′	m	e	m	m	m	o	o	4g+1, 4g′+2	V′B′
21	20	$1'$	V′T′	o	e	m	m	o	m	m	4g+1, 4g′+3	V′T′
22	21		B′T′	m	e	o	o	m	m	m	4g+2, 4g′+3	B′T′
23	19b		S′V′B′T′	m	e	m	m	m	m	m	Three or four out of: 4g, 4g′+1, 4g″+2, 4g‴+3	V′B′T′, S′B′T′, S′V′T′, S′V′B′, S′V′B′T′
24	22a		SV′	m	m	e	m	o	m	o	4g, 4g′+1	SV′
25	23	$1^\dagger$	V′V	o	m	e	m	m	o	m	4g+1, 4g′+1	V′V
26	24		SV	m	o	e	o	m	m	m	4g, 4g′+1	SV
27	22b		S′SV′V	m	m	e	m	m	m	m	Three or four out of: 4g, 4g′, 4g″+1, 4g‴+1	SV′V, S′V′V, S′SV, S′SV′, S′SV′V
28	25a		V′B	m	m	m	e	o	o	m	4g+1, 4g′+2	V′B
29	26	$1'^\dagger$	BT	m	o	o	e	m	m	m	4g+2, 4g′+3	BT
30	27		V′T	o	m	m	e	m	m	o	4g+1, 4g′+3	V′T
31	25b		S′V′BT	m	m	m	e	m	m	m	Three or four out of: 4g, 4g′+1, 4g″+2, 4g‴+3	V′BT, S′BT, S′V′T, S′V′B, S′V′BT
32	28a		VB′	m	m	m	o	e	o	m	4g+1, 4g′+2	VB′
33	29	$\bar{1}'$	VT	o	o	m	m	e	m	m	4g+1, 4g′+3	VT
34	30		B′T	m	m	o	m	e	m	o	4g+2, 4g′+3	B′T
35	28b		S′VB′T	m	m	m	m	e	m	m	Three or four out of: 4g, 4g′+1, 4g″+2, 4g‴+3	VB′T, S′B′T, S′VT, S′VB′, S′VB′T
36	31a		ST′	m	m	m	o	o	e	m	4g, 4g′+3	ST′
37	32	$\bar{1}^\dagger$	T′T	o	m	o	m	m	e	m	4g+3, 4g′+3	T′T
38	33		ST	m	o	m	m	m	e	o	4g, 4g′+3	ST
39	31b		S′ST′T	m	m	m	m	m	e	m	Three or four out of: 4g, 4g′, 4g″+3, 4g‴+3	ST′T, S′T′T, S′ST, S′ST′, S′ST′T
40	34		VB	m	o	m	m	m	o	e	4g+1, 4g′+2	VB
41	35	$\bar{1}'^\dagger$	BT′	m	m	o	m	o	m	e	4g+2, 4g′+3	BT′
42	36		VT′	o	m	m	o	m	m	e	4g+1, 4g′+3	VT′
43	37		S′VBT′	m	m	m	m	m	m	e	Three or four out of: 4g, 4g′+1, 4g″+2, 4g‴+3	VBT′, S′BT′, S′VT′, S′VB, S′VBT′

Table 3 (continued)

New No.	Old No.	SS	Label	Action of							Grades	Multivectors (omitting '+' signs)
				$\bar{1}$	$1'$	$1^{\dagger}$	$1'^{\dagger}$	$\bar{1}'$	$\bar{1}^{\dagger}$	$\bar{1}'^{\dagger}$		
44	38a		SVB′T′	m	m	m	o	m	m	m	Three or four out of: 4g, 4g′+1, 4g″+2, 4g‴+3	VB′T′, SB′T′, SVT′, SVB′, SVB′T′
45	38b		SV′BT′	m	m	m	m	o	m	m	Three or four out of: 4g, 4g′+1, 4g″+2, 4g‴+3	V′BT′, SBT′, SV′T′, SV′B, SV′BT′
46	38c		V′VB′B	m	m	m	m	m	o	m	Three or four out of: 4g+1, 4g′+1, 4g″+2, 4g‴+2	VB′B, V′B′B, V′VB, V′VB′, V′VB′B
47	38d		SV′B′T	m	m	m	m	m	m	o	Three or four out of: 4g, 4g′+1, 4g″+2, 4g‴+3	V′B′T, SB′T, SV′T, SV′B′, SV′B′T
48	39	1	B′BT′T	m	m	o	m	m	m	m	Three or four out of: 4g+2, 4g′+2, 4g″+3, 4g‴+3	BT′T, B′T′T, B′BT, B′BT′, B′BT′T
49	40		SVBT	m	o	m	m	m	m	m	Three or four out of: 4g, 4g′+1, 4g″+2, 4g‴+3	VBT, SBT, SVT, SVB, SVBT
50	41		V′VT′T	o	m	m	m	m	m	m	Three or four out of: 4g+1, 4g′+1, 4g″+3, 4g‴+3	VT′T, V′T′T, V′VT, V′VT′, V′VT′T
51	38e		S′SV′VB′BT′T	m	m	m	m	m	m	m	Varied	All other sums of: S′SV′VB′BT′T